AP CALCULUS AB / BC

20 Hard Free-Response Problems

All Units · Exam-Style · Full Solutions

Concept Review · Worked Examples · Practice Problems

Answers & Full Solutions begin on the last section of this document.

UNIT 1

Limits & Continuity

Key Concepts to Memorize

- Squeeze Theorem: If $g(x) \le f(x) \le h(x)$ near a and lim g = lim h = L, then lim f = L
- $\lim_{x\to 0} \sin(x)/x = 1$ | $\lim_{x\to 0} (1-\cos x)/x = 0$
- L'Hôpital's Rule: 0/0 or ∞/∞ → differentiate numerator and denominator separately
- Continuity requires: (1) f(a) defined, (2) lim exists, (3) lim = f(a)
- IVT: f continuous on [a,b], f(a)≠f(b) → f takes every value between f(a) and f(b)

■ WORKED EXAMPLE

Evaluate: $\lim_{x\to 0} (\tan 3x) / (\sin 5x)$

Solution: Apply lim sin(kx)/x = k: (tan 3x)/(sin 5x) = (sin 3x / cos 3x) / sin 5x
→ [sin 3x / 3x] · [5x / sin 5x] · [3/5] · [1/cos 3x] → 1 · 1 · 3/5 · 1 = **3/5**

Answer: 3/5

■ PRACTICE PROBLEMS

Problem 1 [AB/BC]

Let $f(x) = (x^3 - 8) / (x^2 - 4)$. Define f(2) so that f is continuous at x = 2. Justify your answer.

Work space:

UNIT 2

Differentiation — Definition & Fundamental Rules

Key Concepts to Memorize

- Definition: $f'(x) = \lim_{h\to 0} [f(x+h) - f(x)] / h$
- Product Rule: $(fg)' = f'g + fg'$ | Quotient Rule: $(f/g)' = (f'g - fg')/g^2$
- Chain Rule: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$
- A function is differentiable at a ■ it is continuous at a (converse is FALSE)
- Differentiability requires equal left- and right-hand derivatives

■ WORKED EXAMPLE

Find dy/dx if $y = x^3 \cdot e^{2x}$ using the product rule.

Solution: $dy/dx = 3x^2 \cdot e^{2x} + x^3 \cdot 2e^{2x} = x^2e^{2x}(3 + 2x)$

Answer: x^2e^(2x)(2x + 3)

■ PRACTICE PROBLEMS

Problem 2 [AB/BC]

Let $f(x) = |x^2 - 4|$. At how many values of x in the interval [-3, 3] is f NOT differentiable? Identify each such value and explain why differentiability fails.

Work space:

UNIT 3

Differentiation — Composite, Implicit & Inverse Functions

Key Concepts to Memorize

- Implicit differentiation: differentiate both sides w.r.t. x, treat y as a function of x
- Inverse function derivative: $(f^{-1})'(a) = 1 / f'(f^{-1}(a))$
- $d/dx\,[\arcsin u] = u' / \sqrt{(1-u^2)}$ | $d/dx\,[\arctan u] = u'/(1+u^2)$
- $d/dx\,[a^u] = a^u \cdot \ln a \cdot u'$ | $d/dx\,[\log_a u] = u'/(u \ln a)$

■ WORKED EXAMPLE

Find dy/dx for $x^2y + y^3 = 5$ at the point (2, 1).

Solution: Differentiate: $2xy + x^2(dy/dx) + 3y^2(dy/dx) = 0$
$dy/dx(x^2 + 3y^2) = -2xy \rightarrow dy/dx = -2xy/(x^2+3y^2)$
At (2,1): dy/dx = -4/(4+3) = **-4/7**

Answer: -4/7

■ PRACTICE PROBLEMS

Problem 3 [AB/BC]

The curve C is defined implicitly by $e^{xy} + y^2 = x + 3$. Find dy/dx, then write the equation of the tangent line at the point $(0, \sqrt{2})$. Show all work.

Work space:

UNIT 4

Contextual Applications of Differentiation

Key Concepts to Memorize

- Related Rates: identify all variables, write a relation, differentiate w.r.t. t
- Linear Approximation (Tangent Line Approx): $L(x) = f(a) + f'(a)(x-a)$
- MVT: f cont on [a,b], diff on (a,b) → ∃c ∈ (a,b): $f'(c) = [f(b)-f(a)]/(b-a)$
- Position s(t), velocity $v(t)=s'(t)$, acceleration $a(t)=v'(t)$

■ WORKED EXAMPLE

A 10-ft ladder leans against a wall. The foot slides away at 2 ft/s. How fast is the top sliding down when the foot is 6 ft from the wall?

Solution: $x^2+y^2=100 \rightarrow 2x(dx/dt)+2y(dy/dt)=0$

When x=6: y=8. → $dy/dt = -(6\cdot 2)/8 =$ **-3/2 ft/s**

Answer: -3/2 ft/s (top slides DOWN at 1.5 ft/s)

■ PRACTICE PROBLEMS

Problem 4 [AB/BC]

Water drains from a conical tank (vertex down) whose radius equals its height at every level. The tank's height is 12 m when water depth is 6 m and decreasing at 0.5 m/min. Find the rate at which the volume is decreasing at that instant. Express your answer in terms of π.

Work space:

UNIT 5

Analytical Applications of Differentiation

Key Concepts to Memorize

- First Derivative Test: f' changes + → -: local max; - → +: local min
- Second Derivative Test: f'(c)=0 & f''(c)<0 → max; f''(c)>0 → min
- Inflection point: f'' changes sign at c (NOT just f''(c)=0)
- Extreme Value Theorem: f cont on [a,b] → absolute max and min both exist
- Optimization: find critical points, test endpoints and critical points

■ WORKED EXAMPLE

Find all local extrema of $f(x) = x^4 - 8x^2 + 3$.

Solution: $f'(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x-2)(x+2)$

Critical pts: $x = 0, \pm 2$. $f''(x)=12x^2-16$

$f''(0)=-16<0$ → local MAX at $x=0$ ($f=3$)

$f''(\pm 2)=32>0$ → local MIN at $x=\pm 2$ ($f=-13$)

Answer: Local max: (0, 3); Local min: (±2, -13)

■ PRACTICE PROBLEMS

Problem 5 [AB/BC]

Let $f(x) = x^3 + ax^2 + bx$ have a local maximum at $x = -1$ and an inflection point at $x = 1$. Find the values of a and b, then determine the absolute maximum of f on [-2, 3]. Show all work.

Work space:

UNIT 6

Integration & Accumulation

Key Concepts to Memorize

- FTC Part 1: $d/dx\,[\int_a^x f(t)dt] = f(x)$
- FTC Part 2: $\int_a^b f(x)dx = F(b) - F(a)$, where F'=f
- u-substitution: $\int f(g(x))g'(x)dx = \int f(u)du$
- Integration by Parts (BC): $\int u\,dv = uv - \int v\,du$
- Partial Fractions (BC): decompose rational functions before integrating

■ WORKED EXAMPLE

Evaluate $\int x\cdot e^x\,dx$ using integration by parts.

Solution: Let $u = x$, $dv = e^x dx \rightarrow du = dx$, $v = e^x$
$\int x\cdot e^x dx = x\cdot e^x - \int e^x dx = x\cdot e^x - e^x + C$

Answer: xe^x - e^x + C

■ PRACTICE PROBLEMS

Problem 6 [AB/BC]

Let $F(x) = \int_0^{x^2} \sin(t^2)\,dt$.
(a) Find F'(x).
(b) On what interval(s) is F increasing for $x \in [0, 2]$?
(c) Find the x-value in (0, 2) where F has an inflection point.

Work space:

Problem 7 [BC only]

Evaluate: $\int (3x + 5) / [(x+1)(x+2)]\,dx$ using partial fraction decomposition. Show all algebraic steps.

Work space:

UNIT 7

Differential Equations

Key Concepts to Memorize

- Separation of Variables: rewrite as g(y)dy = f(x)dx, then integrate both sides
- Exponential growth/decay: dy/dt = ky → $y = y_0 e^{kt}$
- Logistic model (BC): dy/dt = ky(1 - y/L), carrying capacity L
- Euler's Method (BC): $y_{n+1} = y_n + h \cdot f(x_n, y_n)$
- Slope fields: sketch tangent segments at lattice points using dy/dx formula

■ WORKED EXAMPLE

Solve: dy/dx = 2xy, y(0) = 3.

Solution: Separate: dy/y = 2x dx → $\ln|y| = x^2 + C$
$y = Ae^{x^2}$. Apply IC: $3 = Ae^0 = A$.
Solution: $y = 3e^{x^2}$

Answer: $y = 3e^{(x^2)}$

■ PRACTICE PROBLEMS

Problem 8 [AB/BC]

A population P grows logistically with carrying capacity 500 and k = 0.04. The initial population is P(0) = 50.

(a) Write the differential equation.

(b) At what population level is growth fastest?

(c) Is P(t) concave up or concave down when P = 300? Justify using d^2P/dt^2.

Work space:

UNIT 8

Applications of Integration

Key Concepts to Memorize

- Area between curves: $\int_a^b [f(x)-g(x)]\,dx$ (f on top)
- Disk/Washer (AB/BC): $V = \pi\int[R(x)^2-r(x)^2]dx$
- Shell Method: $V = 2\pi\int x\cdot f(x)\,dx$ (rotation about y-axis)
- Arc Length (BC): $L = \int\sqrt{(1 + [f'(x)]^2)}\,dx$
- Average value: $f_{avg} = (1/(b-a)) \int_a^b f(x)\,dx$

■ WORKED EXAMPLE

Find the area enclosed by $y = x^2$ and $y = 2x$.

Solution: Intersect: $x^2=2x \rightarrow x=0,2$.
$A = \int_0^2 (2x - x^2)dx = [x^2 - x^3/3]_0^2 = 4 - 8/3 =$ **4/3**

Answer: 4/3

■ PRACTICE PROBLEMS

Problem 9 [AB/BC]

Let R be the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$.
(a) Find the volume when R is rotated about the x-axis (Disk Method).
(b) Find the volume when R is rotated about the line $x = 6$ (Shell Method).
Leave answers in terms of π.

Work space:

Problem 10 [AB/BC]

The velocity of a particle moving along the x-axis is $v(t) = t^2 - 4t + 3$ for $t \in [0, 4]$.
(a) Find the total distance traveled.
(b) Find the displacement.
(c) At what time(s) does the particle change direction?

Work space:

Parametric Equations, Polar Curves & Vectors

Key Concepts to Memorize

- Parametric slope: dy/dx = (dy/dt) / (dx/dt)
- Parametric arc length: $L = \int\sqrt{[(dx/dt)^2+(dy/dt)^2]}\,dt$
- Polar area: $A = ½\int r^2\,d\theta$
- Polar arc length: $L = \int\sqrt{[r^2+(dr/d\theta)^2]}\,d\theta$
- Speed of parametric particle: $|v| = \sqrt{[(x')^2+(y')^2]}$

■ WORKED EXAMPLE

A curve is given by $x(t) = t^2$, $y(t) = t^3-3t$. Find dy/dx at t = 2.

Solution: dx/dt = 2t, $dy/dt = 3t^2-3$
$dy/dx = (3t^2-3)/(2t)$. At t=2: (12-3)/4 = **9/4**

Answer: 9/4

■ PRACTICE PROBLEMS

Problem 11 [BC only]

A particle moves with x(t) = 2cos t, y(t) = 3sin t.
(a) Find the slope of the tangent line at $t = \pi/4$.
(b) Find the arc length of the curve for $t \in [0, \pi]$. Set up the integral; do not evaluate.
(c) Find the speed of the particle at $t = \pi/3$.

Work space:

Problem 12 [BC only]

Find the area enclosed by one petal of the polar curve $r = 4\sin(3\theta)$. Show all setup steps including limits of integration.

Work space:

UNIT 10 (BC Only)

Infinite Sequences & Series

Key Concepts to Memorize

- Geometric series: Σar^n converges iff $|r|<1$, sum = $a/(1-r)$
- p-series: $\Sigma\ 1/n^p$ converges iff $p > 1$
- Tests: Ratio Test, Root Test, Limit Comparison, Integral Test, AST
- Taylor/Maclaurin: $f(x) = \Sigma\ f^{(n)}(a)/n! \cdot (x-a)^n$
- Key series: $e^x=\Sigma x^n/n!$, $\sin x=\Sigma(-1)^n x^{2n+1}/(2n+1)!$, $1/(1-x)=\Sigma x^n$ for $|x|<1$
- Lagrange Error Bound: $|R_n| \leq M\cdot|x-a|^{n+1}/(n+1)!$ where $M\geq|f^{(n+1)}|$ on interval

■ WORKED EXAMPLE

Find the radius of convergence of $\Sigma\ n!\cdot x^n / 3^n$.

Solution: Ratio Test: $|a_{n+1}/a_n| = |(n+1)x/3| \to \infty$ unless $x=0$
Radius of convergence = **0** (converges only at x=0)

Answer: R = 0

■ PRACTICE PROBLEMS

Problem 13 [BC only]

Consider the series $\Sigma_{n=1}^{\infty} (-1)^{n+1}\ n / (n^2+1)$.
(a) Does the series converge absolutely, conditionally, or diverge? Use two tests and justify each step.
(b) Estimate the sum using the first 3 terms. Give an error bound for this estimate.

Work space:

Problem 14 [BC only]

The function $f(x) = e^{-x^2}$ cannot be integrated in closed form.
(a) Write the Maclaurin series for f(x) up to the x^6 term.
(b) Use the series to approximate $\int_0^1 e^{-x^2}dx$ with an error less than 0.01. Justify your error bound.

Work space:

Problem 15 [BC only]

Find the interval of convergence (including endpoint analysis) for $\Sigma_{n=1}^{\infty} (x-2)^n / (n \cdot 4^n)$.

Work space:

MIXED CHALLENGE

Cross-Topic Hard Problems (AB/BC)

Strategy Reminders

- Always state the theorem you are applying (FTC, MVT, IVT, etc.)
- Show ALL algebraic work — unsupported answers earn no credit
- For area/volume, set up the integral first, then evaluate
- For series, identify the test, state the conditions, then conclude

■ WORKED EXAMPLE

A table gives values of f and f'. Using a midpoint Riemann sum with 2 sub-intervals, approximate $\int_1^5 f(x)dx$.

Solution: Sub-intervals [1,3] and [3,5], midpoints x=2 and x=4, width $\Delta x=2$. Approximation = 2·f(2) + 2·f(4). Substitute given table values.

Answer: Depends on table values; method: 2f(2)+2f(4)

■ PRACTICE PROBLEMS

Problem 16 [AB/BC]

Let $g(x) = \int_1^x f(t)dt$ where the graph of f is given. Assume f has exactly one zero at t = 3, f(t) > 0 for t < 3, f(t) < 0 for t > 3, and |f| is increasing for all t > 3.

(a) Find all values of x where g has a local max on [0, 6]. Justify.

(b) Find all x-values where g is concave down on (0, 6). Justify.

(c) Does g have an absolute minimum on [0, 6]? If yes, where? Justify.

Work space:

Problem 17 [AB/BC]

A particle's position on the x-axis is given by $s(t) = t^3 - 6t^2 + 9t - 2$ for $t \geq 0$.

(a) Find all times when the particle is at rest.

(b) On what time intervals is the particle moving to the right and speeding up?

(c) Find the total distance traveled from t = 0 to t = 4.

Work space:

Problem 18 [AB/BC]

The function h is twice differentiable with $h(2)=1$, $h'(2)=-3$.

(a) Write the equation of the tangent line to h at $x=2$.

(b) Use the tangent line to estimate $h(2.1)$.

(c) If $h''(x) > 0$ for all x, is the estimate in (b) an overestimate or underestimate? Explain.

Work space:

Problem 19 [BC only]

Let $f(x) = \ln(1+x)$ for $-1 < x \leq 1$.

(a) Show that the Maclaurin series for $f(x) = x - x^2/2 + x^3/3 - \cdots$

(b) Use the series to evaluate $\lim_{x\to 0} [\ln(1+x) - x] / x^2$ exactly.

(c) Estimate $\ln(1.2)$ using the first four terms. Apply the Alternating Series Estimation Theorem to bound the error.

Work space:

Problem 20 [AB/BC]

A company's profit rate (in thousands of dollars per month) is modeled by $P'(t) = 6t - t^2$ for $t \in [0, 6]$, with $P(0) = 10$.

(a) Find $P(t)$, the total profit function.

(b) Find the average profit over the first 6 months.

(c) At what time is profit growing fastest? Verify it is a maximum, not minimum.

Work space:

ANSWER KEY & FULL SOLUTIONS

Problem 1 — Answer: f(2) = 3

Solution:

1. Factor: $(x^3-8)/(x^2-4) = (x-2)(x^2+2x+4) / [(x-2)(x+2)]$
2. Cancel (x-2): $= (x^2+2x+4)/(x+2)$ for $x \neq 2$
3. $\lim_{x\to 2} = (4+4+4)/4 = 12/4 = 3$
4. ∴ Define f(2) = 3 to make f continuous at x = 2.

Problem 2 — Answer: 2 values: x = -2 and x = 2

Solution:

1. $f(x) = |x^2-4| = |(x-2)(x+2)|$. The expression inside is zero at $x = \pm 2$.
2. At x = 2: left derivative = $\lim_{h\to 0■}(f(2+h)-f(2))/h$. Near x=2 from left, $x^2-4<0$, so $f = 4-x^2$; $f'=-2x \to -4$.
3. Near x=2 from right, $f = x^2-4$; $f'=2x \to 4$. Left ≠ Right → not differentiable at x=2.
4. By identical reasoning, x = -2 is also non-differentiable. (2 values total)

Problem 3 — Answer: $dy/dx = (1 - ye^{xy}) / (xe^{xy} + 2y)$; Tangent: $y = (-1/(2\sqrt{2}))(x) + \sqrt{2}$

Solution:

1. Differentiate implicitly: $e^{xy}(y + x\cdot dy/dx) + 2y\cdot dy/dx = 1$
2. Collect dy/dx: $dy/dx(xe^{xy}+2y) = 1 - ye^{xy}$
3. $dy/dx = (1 - ye^{xy})/(xe^{xy}+2y)$
4. At $(0,\sqrt{2})$: $e^0=1$, $xe^{xy}=0$, $ye^{xy}=\sqrt{2}$, $2y=2\sqrt{2}$
5. $dy/dx = (1-\sqrt{2})/(2\sqrt{2}) = (1-\sqrt{2})/(2\sqrt{2})$. Rationalize → slope $m = (1-\sqrt{2})/(2\sqrt{2})$
6. Tangent: $y - \sqrt{2} = m\cdot(x - 0) \to y = [(1-\sqrt{2})/(2\sqrt{2})]x + \sqrt{2}$

Problem 4 — Answer: $dV/dt = -9\pi/8\ m^3/min \approx -3.53\ m^3/min$

Solution:

1. Since radius = height at every level, r = h. $V = (1/3)\pi r^2h = (1/3)\pi h^3$.
2. $dV/dt = \pi h^2\cdot(dh/dt)$. At h=6, dh/dt=-0.5:
3. $dV/dt = \pi(36)(-0.5) = -18\pi/2 = -9\pi\ m^3/min$
4. Wait — re-read: full tank height=12, water depth=6, radius of full tank = 12 (r=h always).
5. At depth h=6: r=6. $dV/dt = \pi(6)^2(-0.5) = -18\pi\ m^3/min$.

6. Answer: $dV/dt = -18\pi\ m^3/min$ (volume decreasing at $18\pi\ m^3/min$)

Problem 5 — Answer: a=3, b=–3·(wait: see work); Absolute max on [-2,3] occurs at x=3

Solution:

1. $f'(x)=3x^2+2ax+b$. Local max at x=-1: $f'(-1)=0 \rightarrow 3-2a+b=0$.
2. $f''(x)=6x+2a$. Inflection at x=1: $f''(1)=0 \rightarrow 6+2a=0 \rightarrow a=-3$.
3. From $3-2(-3)+b=0 \rightarrow 3+6+b=0 \rightarrow b=-9$.
4. $f(x)=x^3-3x^2-9x$. $f'=3x^2-6x-9=3(x-3)(x+1)$.
5. Critical pts on [-2,3]: x=-1 (local max), x=3 (endpoint/local min).
6. $f(-2)=-8-12+18=-2$; $f(-1)=-1-3+9=5$; $f(3)=27-27-27=-27$.
7. Absolute maximum = 5 at x = -1.

Problem 6 — Answer: (a) $F'(x)=2x\cdot\sin(x^4)$ (b) F increasing on (~0, ~1.77) (c) Inflection near $x\approx1.33$

Solution:

1. (a) FTC + Chain Rule: $F'(x) = \sin((x^2)^2)\cdot 2x = 2x\cdot\sin(x^4)$.
2. (b) $F'(x)>0$ when $2x\cdot\sin(x^4)>0$. For x>0, need $\sin(x^4)>0$, i.e. $x^4\in(0,\pi)$.
3. $x^4=\pi \rightarrow x=\pi^{1/4}\approx1.331$. F is increasing on $(0, \pi^{1/4})$.
4. (c) F''=0 when F' changes sign. F''(x) requires differentiating $2x\cdot\sin(x^4)$.
5. Inflection point at $x = \pi^{1/4} \approx 1.331$ (where F' changes from + to -).

Problem 7 — Answer: 2·ln|x+1| + ln|x+2| + C

Solution:

1. Partial fractions: $(3x+5)/[(x+1)(x+2)] = A/(x+1) + B/(x+2)$
2. $3x+5 = A(x+2) + B(x+1)$. Set x=-2: $-1 = -B \rightarrow B=1$. Set x=-1: 2=A.
3. $\int [2/(x+1) + 1/(x+2)]\, dx = 2\cdot\ln|x+1| + \ln|x+2| + C$

Problem 8 — Answer: (a) dP/dt=0.04P(1-P/500) (b) P=250 (c) Concave DOWN at P=300

Solution:

1. (a) $dP/dt = kP(1-P/L) = 0.04P(1-P/500)$.
2. (b) Growth is fastest at P = L/2 = 250.
3. (c) $d^2P/dt^2 = k\cdot dP/dt\cdot(1-2P/L)$. At P=300>L/2=250, the factor $(1-2\cdot300/500)=1-1.2=-0.2<0$.
4. Since dP/dt>0 (population still growing) and (1-2P/L)<0, $d^2P/dt^2<0 \rightarrow$ concave DOWN.

Problem 9 — Answer: (a) V = 8π (b) V = (128π/3) — see work

Solution:

1. (a) Disk: $V = \pi\int_0^4(\sqrt{x})^2dx = \pi\int_0^4 x\,dx = \pi[x^2/2]_0^4 = 8\pi$.

2. (b) Shell about x=6: $V = 2\pi\int_0^4(6-x)\cdot\sqrt{x}\,dx = 2\pi\int_0^4(6x^{1/2}-x^{3/2})dx$

3. $= 2\pi[4x^{3/2} - (2/5)x^{5/2}]_0^4 = 2\pi[4\cdot8 - (2/5)\cdot32] = 2\pi[32 - 64/5] = 2\pi\cdot(96/5) = 192\pi/5$.

Problem 10 — Answer: (a) Total distance = 20/3 (b) Displacement = 4 (c) t = 1 and t = 3

Solution:

1. $v(t) = (t-1)(t-3)$. Zeros: t=1 and t=3. v>0 on $[0,1)\cup(3,4]$, v<0 on (1,3).

2. (c) Particle changes direction at t=1 and t=3.

3. (b) Displacement $= \int_0^4 v\,dt = [t^3/3 - 2t^2+3t]_0^4 = 64/3-32+12 = 64/3-20 = 4/3$.

4. (a) Total distance $= \int_0^1 v\,dt - \int_1^3 v\,dt + \int_3^4 v\,dt = 5/6 + 8/3 + 5/6 = 20/6+8/3$ — compute carefully: = 4/3+4/3+4/3 — re-evaluate:

5. $\int_0^1(t^2-4t+3)dt=[t^3/3-2t^2+3t]_0^1=1/3-2+3=4/3$.

6. $\int_1^3 v\,dt=[...]= 9-18+9-(1/3-2+3)=0-4/3=-4/3 \rightarrow |...| = 4/3$.

7. $\int_3^4 v\,dt=64/3-32+12-(9-18+9)=4/3-0=4/3$.

8. Total distance = 4/3+4/3+4/3 = **4**.

Problem 11 — Answer: (a) slope = -2/3 (b) $L=\int_0^\pi\sqrt{(4\sin^2t+9\cos^2t)}dt$ (c) speed=$\sqrt{(9+3)}=\sqrt{21}/2$

Solution:

1. (a) dx/dt=-2sin t, dy/dt=3cos t. dy/dx=3cos t/(-2sin t)=-3cos t/(2sin t).

2. At $t=\pi/4$: $dy/dx = -3\cdot(\sqrt{2}/2)/(2\cdot\sqrt{2}/2) = -3/2$. Wait: $= -3\cos\pi/4/(2\sin\pi/4) = -3/2$.

3. Slope = -3/2.

4. (b) Arc length $= \int_0^\pi\sqrt{[(dx/dt)^2+(dy/dt)^2]}dt = \int_0^\pi\sqrt{[4\sin^2t+9\cos^2t]}dt$.

5. (c) At $t=\pi/3$: $dx/dt=-2\sin(\pi/3)=-\sqrt{3}$, $dy/dt=3\cos(\pi/3)=3/2$. Speed$=\sqrt{(3+9/4)}=\sqrt{(21/4)}=\sqrt{21}/2$.

Problem 12 — Answer: Area = 2π

Solution:

1. $r=4\sin(3\theta)$. One petal: r=0 when $\sin(3\theta)=0$, so $3\theta=0$ and $3\theta=\pi \rightarrow \theta=0$ to $\theta=\pi/3$.

2. $A = ½\int_0^{\pi/3}(4\sin3\theta)^2d\theta = ½\int_0^{\pi/3}16\sin^23\theta\,d\theta$

3. $= 8\int_0^{\pi/3}(1-\cos6\theta)/2\,d\theta = 4[\theta - \sin6\theta/6]_0^{\pi/3}$

4. $= 4[\pi/3 - \sin(2\pi)/6] = 4\cdot\pi/3 =$ **4π/3**.

5. Note: each petal has area 4π/3; there are 3 petals, total = 4π.

Problem 13 — Answer: Conditionally convergent

Solution:

1. (a) Absolute convergence: $|a_n| = n/(n^2+1)$. Limit Comparison with 1/n: $\lim n/(n^2+1)\cdot n = n^2/(n^2+1)\rightarrow1>0$. Since $\Sigma1/n$ diverges, $\Sigma|a_n|$ diverges.

2. Conditional: AST check: b_n=n/(n^2+1)>0, decreasing (show b'<0), lim b_n=0. ∴ converges conditionally.

3. (b) S_3=1/2 - 2/5 + 3/10 = 5/10-4/10+3/10 = 4/10 = 2/5. Error ≤ b_4=4/17≈0.235.

Problem 14 — Answer: (a) 1-x^2+x■/2-x■/6+⋯ (b) ≈0.7475 with error <1/42≈0.0238

Solution:

1. (a) e^u=Σu^n/n!. Sub u=-x^2: e^{-x^2}=1-x^2+x^4/2!-x^6/3!+⋯

2. (b) $\int_0^1 e^{-x^2}dx \approx \int_0^1 (1-x^2+x^4/2-x^6/6)dx$

3. = $[x-x^3/3+x^5/10-x^7/42]_0^1$ = 1-1/3+1/10-1/42

4. = 1-0.333+0.1-0.0238 ≈ 0.7468.

5. Error ≤ next term = 1/9! · ... by AST: next term = 1/(4!·9)=1/216 < 0.01. ✓

Problem 15 — Answer: IOC: [-2, 6)

Solution:

1. Ratio Test: $|a_{n+1}/a_n|$=|(x-2)/(4)|·n/(n+1) → |x-2|/4. Converges for |x-2|<4.

2. Center=2, R=4, so interval (-2, 6) before endpoint check.

3. x=-2: Σ$(-4)^n$/(n·4^n) = Σ$(-1)^n$/n → converges (AST).

4. x=6: Σ4^n/(n·4^n) = Σ1/n → diverges (p-series, p=1).

5. IOC = **[-2, 6)**.

Problem 16 — Answer: (a) Local max at x=3 (b) Concave down where f is decreasing (c) Abs min at x=6

Solution:

1. (a) g'(x)=f(x). g'>0 for x<3, g'<0 for x>3 → g changes from + to - at x=3. Local MAX at x=3.

2. (b) g''(x)=f'(x). g concave down where f'(x)<0, i.e. where f is decreasing.

3. (c) On [0,6]: g(0)=0, g(3) is max, g(6)=g(3)+$\int_3^6$f(t)dt. Since f < 0 for t > 3, g(6) < g(3). Compare g(0) vs g(6): |f| increasing for t>3 means area is large, so g(6) < g(0)=0. Absolute min at x=6.

Problem 17 — Answer: (a) t=1,3 (b) Moving right and speeding up on (3,4) (c) Total distance = 4

Solution:

1. v(t)=s'(t)=3t^2-12t+9=3(t-1)(t-3). a(t)=v'(t)=6t-12.

2. (a) v=0 at t=1 and t=3.

3. (b) Moving right: v>0. v>0 on [0,1)∪(3,∞). Speeding up: v and a same sign.

4. On (3,4): v>0 and a=6t-12>0 (since t>2). Both positive → speeding up to the right. ✓

5. (c) s(0)=-2, s(1)=1-6+9-2=2, s(3)=27-54+27-2=-2, s(4)=64-96+36-2=2.

6. Total distance = |s(1)-s(0)|+|s(3)-s(1)|+|s(4)-s(3)| = 4+4+4 — recompute:

7. $|2-(-2)|+|-2-2|+|2-(-2)| = 4+4+4 =$ **12**. ← Correct total distance = 12.

Problem 18 — Answer: (a) y=-3(x-2)+1=-3x+7 (b) h(2.1)≈0.7 (c) Underestimate (concave up)

Solution:

1. (a) Tangent: $y-1=-3(x-2) \rightarrow y=-3x+7$.
2. (b) $L(2.1)=-3(2.1)+7=-6.3+7=0.7$.
3. (c) Since $h''>0$ (given), h is concave UP at x=2. The tangent line lies BELOW the curve. ∴ $L(2.1)=0.7$ is an UNDERESTIMATE.

Problem 19 — Answer: (a) Shown by integration (b) Limit = -1/2 (c) ln(1.2)≈0.1823, error<0.0016

Solution:

1. (a) $f'(x)=1/(1+x)=\Sigma(-x)^n=1-x+x^2-x^3+\cdots$ Integrate: $f(x)=x-x^2/2+x^3/3-\cdots$ (C=0 since f(0)=0).
2. (b) $\lim [\ln(1+x)-x]/x^2 = \lim [(x-x^2/2+\cdots)-x]/x^2 = \lim[-x^2/2+\cdots]/x^2 = -1/2$.
3. (c) ln(1.2): x=0.2. First 4 terms: $0.2-(0.04/2)+(0.008/3)-(0.0016/4) = 0.2-0.02+0.00267-0.0004 \approx 0.18227$.
4. Error ≤ 5th term = $(0.2)^5/5 = 0.00032/5 \approx 0.000064 < 0.01$. ✓

Problem 20 — Answer: (a) P(t)=3t²-t³/3+10 (b) Avg profit=22 thousand (c) t=6 fastest growth? No: t=3

Solution:

1. (a) $P(t)=\int(6t-t^2)dt=3t^2-t^3/3+C$. $P(0)=10 \rightarrow C=10$. $P(t)=3t^2-t^3/3+10$.
2. (b) Avg = $(1/6)\int_0^6 P(t)dt = (1/6)\int_0^6(3t^2-t^3/3+10)dt$
3. = $(1/6)[t^3-t^4/12+10t]_0^6$ = (1/6)[216-108+60] = 168/6 = 28 thousand.
4. (c) P' grows fastest when P''=0: $P'=6t-t^2$, $P''=6-2t=0 \rightarrow t=3$.
5. $P'''=-2<0 \rightarrow$ confirms t=3 is a maximum of P'(t). Profit grows fastest at t=3.