COUNTING & COMBINATORICS

Exam-Style Practice — 20 Problems

Subject:	Mathematics — Probability & Combinatorics
Level:	High School / University Entrance
Questions:	20 (Short-answer / Show-your-work)
Topics:	Counting Principle · Permutations · Combinations · Stars & Bars · Inclusion-Exclusion

Instructions: Show all work. Unsupported answers receive no credit.

SECTION 1 — CONCEPTS & KEY FORMULAS

1. Fundamental Counting Principle

• If event A can occur in m ways and event B can occur in n ways, then A AND B together = m × n ways.

• If event A can occur in m ways and event B can occur in n ways (mutually exclusive), then A OR B = m + n ways.

Formulas:

Product Rule: m × n

Sum Rule: m + n

Worked Example:
Q: A restaurant offers 3 soups and 5 main courses. How many different meals (1 soup + 1 main) are possible?
A: 3 × 5 = 15 meals

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2. Permutations

• A permutation is an arrangement of objects where ORDER matters.

• P(n, r) = n! / (n-r)! — choose r from n distinct objects.

• Circular permutation of n objects = (n-1)!

• Permutation with repetition: if n objects have groups of p, q, r identical items: n! / (p! q! r!)

Formulas:

nPr = n! / (n-r)!

Circular: (n-1)!

With repetition: n! / (p!q!r!...)

Worked Example:
Q: How many ways can 5 students be arranged in a line?
A: 5! = 120

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3. Combinations

- A combination is a selection where ORDER does NOT matter.
- C(n, r) = n! / (r!(n-r)!) — also written nCr or C(n,r).
- Key identity: C(n, r) = C(n, n-r).
- Pascal's identity: C(n, r) = C(n-1, r-1) + C(n-1, r).

Formulas:

nCr = n! / (r!(n-r)!)

C(n,r) = C(n, n-r)

C(n,r) = C(n-1,r-1) + C(n-1,r)

Worked Example:

Q: How many ways can 3 students be selected from a group of 8?

A: C(8,3) = 8!/(3!·5!) = 56

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4. Permutations vs Combinations — Decision Rule

- Ask: does the ORDER of selection matter?
- YES → Permutation (P)
- NO → Combination (C)
- Relationship: P(n,r) = C(n,r) × r!

Formulas:

nPr = nCr × r!

Worked Example:

Q: Selecting a president and vice-president from 10 people (order matters) vs choosing a committee of 2 (order doesn't matter).

A: President/VP: P(10,2)=90 ; Committee: C(10,2)=45

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5. Stars and Bars / Combinations with Repetition

• Choosing r items from n types with repetition allowed (order doesn't matter):

• H(n, r) = C(n+r-1, r)

• This is equivalent to distributing r identical balls into n distinct boxes.

Formulas:

H(n,r) = C(n+r-1, r)

Worked Example:

Q: How many ways can you choose 3 fruits from 4 types (apple, banana, orange, grape) if repetition is allowed?

A: H(4,3) = C(6,3) = 20

SECTION 2 — EXAM PROBLEMS

Solve each problem. Show your reasoning clearly. Answers are collected at the end.

Problem 1

[Fundamental Counting Principle]

A password consists of 2 distinct letters (from A–Z) followed by 3 distinct digits (from 0–9). How many different passwords can be formed?

Answer:

Problem 2

[Sum Rule]

A student must choose one elective from either the Science department (4 courses) or the Arts department (6 courses). How many choices does the student have?

Answer:

Problem 3

[Basic Permutation]

Eight runners compete in a race. In how many ways can the gold, silver, and bronze medals be awarded?

Answer:

Problem 4

[Permutation with Condition]

How many 5-digit numbers can be formed using the digits 1, 2, 3, 4, 5 (each used exactly once) such that the number is even?

Answer:

Problem 5

[Circular Permutation]

Six people sit around a circular table. In how many distinct ways can they be seated if two specific people, A and B, must always sit next to each other?

Answer:

Problem 6

[Permutation with Repetition]

How many distinct arrangements are there of the letters in the word MISSISSIPPI?

Answer:

Problem 7

[Basic Combination]

A committee of 4 is to be chosen from 10 people (6 men and 4 women). How many committees contain at least 1 woman?

Answer:

Problem 8

[Combination — Two Groups]

From 5 math books and 4 science books, how many ways can you choose 3 math books and 2 science books?

Answer:

Problem 9

[Combination Identity]

Find the value of n if C(n, 2) = 45.

Answer:

Problem 10

[Combination with Restriction]

A team of 5 is selected from 8 players. Player A refuses to play if Player B is on the team. How many valid teams are there?

Answer:

SECTION 2 — EXAM PROBLEMS (continued)

Problem 11

[Combinations with Repetition]

How many non-negative integer solutions does the equation x + y + z = 10 have?

Answer:

Problem 12

[Combinations with Repetition — Bounded]

How many non-negative integer solutions does x + y + z = 10 have where x ≥ 2, y ≥ 1, z ≥ 0?

Answer:

Problem 13

[Mixed — Permutation & Combination]

A club has 12 members. They must elect a president, a secretary, and then choose a 3-person social committee (none of the officers can be on the committee). In how many ways can this be done?

Answer:

Problem 14

[Grid Paths (Combination Application)]

On a 5×4 grid (5 columns, 4 rows of squares), how many shortest paths exist from the bottom-left corner to the top-right corner, moving only right or up?

Answer:

Problem 15

[Pascal's Triangle / Binomial]

Compute: C(8,0) + C(8,1) + C(8,2) + ... + C(8,8).

Answer:

Problem 16

[Permutation — Fixed Positions]

5 boys and 3 girls stand in a line. In how many ways can they be arranged so that no two girls are adjacent?

Answer:

Problem 17

[Subset Counting]

How many subsets of {1, 2, 3, 4, 5, 6, 7} have exactly 3 elements and contain the element 4?

Answer:

Problem 18

[Complementary Counting]

Six cards labeled 1–6 are dealt one to each of 6 players. In how many ways can the cards be distributed so that NO player receives the card matching their own number? (Derangements)

Answer:

Problem 19

[Dividing into Groups]

In how many ways can 9 students be divided into 3 groups of 3 if the groups are unlabeled (indistinguishable)?

Answer:

Problem 20

[Advanced Counting — Inclusion-Exclusion]

How many integers from 1 to 100 are divisible by 2, 3, or 5?

Answer:

SECTION 3 — ANSWER KEY & FULL SOLUTIONS

Compare your work carefully. Focus on the method, not just the final number.

Problem 1

[Fundamental Counting Principle]

A password consists of 2 distinct letters (from A–Z) followed by 3 distinct digits (from 0–9). How many different passwords can be formed?

Answer: $P(26,2) \times P(10,3) = 650 \times 720 = 468{,}000$

Solution:

Step 1: Choose 2 distinct letters in order from 26: $P(26,2) = 26\times25 = 650$.

Step 2: Choose 3 distinct digits in order from 10: $P(10,3) = 10\times9\times8 = 720$.

Step 3: Multiply (product rule): $650 \times 720 = 468{,}000$.

Problem 2

[Sum Rule]

A student must choose one elective from either the Science department (4 courses) or the Arts department (6 courses). How many choices does the student have?

Answer: $4 + 6 = 10$

Solution:

Since the student picks from Science OR Arts (mutually exclusive), apply the Sum Rule: $4 + 6 = 10$ choices.

Problem 3

[Basic Permutation]

Eight runners compete in a race. In how many ways can the gold, silver, and bronze medals be awarded?

Answer: $P(8,3) = 8\times7\times6 = 336$

Solution:

Order matters (gold ≠ silver ≠ bronze), so use permutation.

$P(8,3) = 8!/(8-3)! = 8\times7\times6 = 336$.

Problem 4

[Permutation with Condition]

How many 5-digit numbers can be formed using the digits 1, 2, 3, 4, 5 (each used exactly once) such that the number is even?

Answer: 48

Solution:

For the number to be even, the last digit must be 2 or 4 (2 choices).

The remaining 4 positions are filled with the remaining 4 digits in $4! = 24$ ways.

Total = $2 \times 24 = 48$.

Problem 5

[Circular Permutation]

Six people sit around a circular table. In how many distinct ways can they be seated if two specific people, A and B, must always sit next to each other?

Answer: $2 \times 4! = 48$

Solution:

Treat A and B as one block → 5 units around a circle: $(5-1)! = 24$ ways.

A and B can switch within the block: $\times 2 = 48$.

Problem 6

[Permutation with Repetition]

How many distinct arrangements are there of the letters in the word MISSISSIPPI?

Answer: 34,650

Solution:

MISSISSIPPI has 11 letters: M×1, I×4, S×4, P×2.

Arrangements = $11! / (1!\ 4!\ 4!\ 2!) = 39{,}916{,}800 / 1152 = 34{,}650$.

Problem 7

[Basic Combination]

A committee of 4 is to be chosen from 10 people (6 men and 4 women). How many committees contain at least 1 woman?

Answer: $C(10,4) - C(6,4) = 210 - 15 = 195$

Solution:

Total committees = $C(10,4) = 210$.

All-male committees = $C(6,4) = 15$.

At least 1 woman = $210 - 15 = 195$.

Problem 8

[Combination — Two Groups]

From 5 math books and 4 science books, how many ways can you choose 3 math books and 2 science books?

Answer: $C(5,3) \times C(4,2) = 10 \times 6 = 60$

Solution:

Choose 3 from 5 math: $C(5,3) = 10$.

Choose 2 from 4 science: $C(4,2) = 6$.

Total = $10 \times 6 = 60$.

Problem 9

[Combination Identity]

Find the value of n if $C(n, 2) = 45$.

Answer: $n = 10$

Solution:

$C(n,2) = n(n-1)/2 = 45 \rightarrow n(n-1) = 90 \rightarrow n^2 - n - 90 = 0$.

Factor: $(n-10)(n+9) = 0 \rightarrow n = 10$ (since $n > 0$).

Problem 10

[Combination with Restriction]

A team of 5 is selected from 8 players. Player A refuses to play if Player B is on the team. How many valid teams are there?

Answer: $C(8,5) - C(6,3) = 56 - 20 = 36$

Solution:

Total = $C(8,5) = 56$.

Invalid (both A and B on team): fix A & B, choose 3 from remaining 6 $\rightarrow C(6,3) = 20$.

Valid = $56 - 20 = 36$.

SECTION 3 — ANSWER KEY (continued)

Problem 11

[Combinations with Repetition]

How many non-negative integer solutions does the equation $x + y + z = 10$ have?

Answer: C(12, 2) = 66

Solution:

This is H(3,10) = C(3+10-1, 10) = C(12,2) = 66.

(Stars-and-bars: distribute 10 identical stars into 3 bins.)

Problem 12

[Combinations with Repetition — Bounded]

How many non-negative integer solutions does $x + y + z = 10$ have where $x \geq 2$, $y \geq 1$, $z \geq 0$?

Answer: C(9, 2) = 36

Solution:

Substitute $x' = x-2$, $y' = y-1$ (both ≥ 0).

New equation: $x' + y' + z = 10 - 2 - 1 = 7$.

Solutions = C(7+2, 2) = C(9,2) = 36.

Problem 13

[Mixed — Permutation & Combination]

A club has 12 members. They must elect a president, a secretary, and then choose a 3-person social committee (none of the officers can be on the committee). In how many ways can this be done?

Answer: P(12,2) × C(10,3) = 132 × 120 = 15,840

Solution:

Elect president & secretary (order matters): P(12,2) = 12×11 = 132.

10 members remain; choose 3 for committee: C(10,3) = 120.

Total = 132 × 120 = 15,840.

Problem 14

[Grid Paths (Combination Application)]

On a 5×4 grid (5 columns, 4 rows of squares), how many shortest paths exist from the bottom-left corner to the top-right corner, moving only right or up?

Answer: C(9, 4) = 126

Solution:

A shortest path needs 5 moves right (R) and 4 moves up (U) = 9 total moves.

Choose which 4 of the 9 steps are 'up': C(9,4) = 126.

Problem 15

[Pascal's Triangle / Binomial]

Compute: C(8,0) + C(8,1) + C(8,2) + ... + C(8,8).

Answer: 2^8 = 256

Solution:

By the Binomial Theorem, the sum of all C(n,k) for k = 0..n equals 2^n.

Here n = 8, so the sum = 2^8 = 256.

Problem 16

[Permutation — Fixed Positions]

5 boys and 3 girls stand in a line. In how many ways can they be arranged so that no two girls are adjacent?

Answer: P(6,3) × 5! = 120 × 120 = 14,400

Solution:

First arrange 5 boys: 5! = 120 ways. This creates 6 gaps (before, between, after boys).

Place 3 girls in 3 of these 6 gaps (order matters): P(6,3) = 6×5×4 = 120.

Total = 120 × 120 = 14,400.

Problem 17

[Subset Counting]

How many subsets of {1, 2, 3, 4, 5, 6, 7} have exactly 3 elements and contain the element 4?

Answer: C(6, 2) = 15

Solution:

Element 4 is already chosen. Choose the remaining 2 elements from the other 6.

C(6,2) = 15.

Problem 18

[Complementary Counting]

Six cards labeled 1–6 are dealt one to each of 6 players. In how many ways can the cards be distributed so that NO player receives the card matching their own number? (Derangements)

Answer: 265

Solution:

The number of derangements of n items is D(n) = n! × Σ((-1)^k / k!) for k=0..n.

D(6) = 720 × (1 - 1 + 1/2 - 1/6 + 1/24 - 1/120 + 1/720)

= 720 × (53/144) = 265.

Problem 19

[Dividing into Groups]

In how many ways can 9 students be divided into 3 groups of 3 if the groups are unlabeled (indistinguishable)?

Answer: C(9,3)×C(6,3)×C(3,3) / 3! = 280

Solution:

Label the groups temporarily:

Group A: C(9,3) = 84, Group B: C(6,3) = 20, Group C: C(3,3) = 1.

Divide by 3! = 6 (groups are identical):

84 × 20 × 1 / 6 = 1680 / 6 = 280.

Problem 20

[Advanced Counting — Inclusion-Exclusion]

How many integers from 1 to 100 are divisible by 2, 3, or 5?

Answer: 74

Solution:

Let A = mult of 2: |A| = 50. B = mult of 3: |B| = 33. C = mult of 5: |C| = 20.

|A∩B| = mult of 6 = 16. |A∩C| = mult of 10 = 10. |B∩C| = mult of 15 = 6.

|A∩B∩C| = mult of 30 = 3.

By inclusion-exclusion:

|A□B□C| = 50+33+20 - 16-10-6 + 3 = 74.