

# Trigonometry

Advanced Practice — 20 Examination-Style Problems

Concepts · Memorisation · Worked Examples · Practice Problems

Answers & Full Solutions on final pages

## CONCEPTS & WORKED EXAMPLES

### Unit 1 Angle Measure & the Unit Circle

#### Key Formulas

Arc length:  $s = r \cdot \theta$  ( $\theta$  in radians)

Sector area:  $A = (1/2) \cdot r^2 \cdot \theta$

Degree  $\leftrightarrow$  Radian:  $\theta(\text{rad}) = \theta(\text{deg}) \cdot (\pi/180)$

Linear speed:  $v = r \cdot \omega$ , Angular speed:  $\omega = \theta/t$

#### Worked Example

Example: A wheel of radius 4 cm turns through  $3\pi/2$  rad. Find arc length.

**Solution:**  $s = 4 \cdot (3\pi/2) = 6\pi \text{ cm} \approx 18.85 \text{ cm}$

#### Unit Circle — Exact Values to Memorise

$\sin 0=0$ ,  $\sin(\pi/6)=1/2$ ,  $\sin(\pi/4)=\sqrt{2}/2$ ,  $\sin(\pi/3)=\sqrt{3}/2$ ,  $\sin(\pi/2)=1$

$\cos 0=1$ ,  $\cos(\pi/6)=\sqrt{3}/2$ ,  $\cos(\pi/4)=\sqrt{2}/2$ ,  $\cos(\pi/3)=1/2$ ,  $\cos(\pi/2)=0$

$\tan = \sin/\cos$  (undefined where  $\cos=0$ )

#### Worked Example

Example: Find the exact value of  $\tan(7\pi/6)$ .

**Solution:**  $\tan(7\pi/6) = \tan(\pi + \pi/6) = \tan(\pi/6) = \sqrt{3}/3$

### Unit 2 Graphs of Trigonometric Functions

## Amplitude, Period, Phase Shift

$$y = A \cdot \sin(Bx + C) + D$$

$$\text{Amplitude} = |A|, \text{Period} = 2\pi/|B|$$

$$\text{Phase shift} = -C/B \text{ (left if negative, right if positive)}$$

$$\text{Vertical shift} = D$$

## Worked Example

Example: State amplitude, period, phase shift of  $y = 3\sin(2x - \pi/4)$ .

**Solution:** Amp=3, Period= $\pi$ , Phase shift =  $\pi/8$  to the right

## Key Graph Properties

$$\sin \text{ is odd: } \sin(-x) = -\sin(x)$$

$$\cos \text{ is even: } \cos(-x) = \cos(x)$$

$$\text{Zeros of } \sin: x = n\pi; \text{ Zeros of } \cos: x = \pi/2 + n\pi \text{ (n integer)}$$

$$\text{Range of } \sin \text{ and } \cos: [-1, 1]; \tan: \text{all reals except } x = \pi/2 + n\pi$$

## Worked Example

Example: Find all zeros of  $y = \cos(2x)$  on  $[0, 2\pi]$ .

**Solution:**  $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

## Unit 3 Trigonometric Identities

### Fundamental Identities

$$\text{Pythagorean: } \sin^2x + \cos^2x = 1$$

$$1 + \tan^2x = \sec^2x$$

$$1 + \cot^2x = \csc^2x$$

$$\text{Reciprocal: } \csc x = 1/\sin x, \sec x = 1/\cos x, \cot x = \cos x/\sin x$$

$$\text{Quotient: } \tan x = \sin x/\cos x$$

## Worked Example

Example: Simplify  $(1 - \sin^2x)/\cos x$ .

**Solution:**  $(1 - \sin^2x)/\cos x = \cos^2x/\cos x = \cos x$

## Sum, Difference & Double-Angle

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A+B) = (\tan A + \tan B)/(1 - \tan A \cdot \tan B)$$

$$\sin(2A) = 2\sin A \cos A$$

$$\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan(2A) = 2\tan A/(1 - \tan^2 A)$$

### Worked Example

Example: Evaluate  $\sin(75^\circ)$  exactly using sum formula.

**Solution:**  $\sin(45^\circ + 30^\circ) = (\sqrt{6} + \sqrt{2})/4$

### Half-Angle & Product-to-Sum

$$\sin(A/2) = \pm \sqrt{(1 - \cos A)/2}$$

$$\cos(A/2) = \pm \sqrt{(1 + \cos A)/2}$$

$$\tan(A/2) = \sin A/(1 + \cos A) = (1 - \cos A)/\sin A$$

$$\sin A \cos B = (1/2)[\sin(A+B) + \sin(A-B)]$$

$$\cos A \cos B = (1/2)[\cos(A-B) + \cos(A+B)]$$

### Worked Example

Example: Find  $\cos(\pi/8)$  exactly.

**Solution:**  $\cos(\pi/8) = \sqrt{(1 + \cos(\pi/4))/2} = \sqrt{(2 + \sqrt{2})/4} = \sqrt{2 + \sqrt{2}}/2$

## Unit 4 Inverse Trigonometric Functions

### Domains & Ranges

$\arcsin x$ : domain  $[-1, 1]$ , range  $[-\pi/2, \pi/2]$

$\arccos x$ : domain  $[-1, 1]$ , range  $[0, \pi]$

$\arctan x$ : domain all reals, range  $(-\pi/2, \pi/2)$

Composition:  $\sin(\arcsin x) = x$  for  $x$  in  $[-1, 1]$

$\arcsin(\sin x) = x$  only if  $x$  in  $[-\pi/2, \pi/2]$

### Worked Example

Example: Evaluate  $\cos(\arctan(3/4))$ .

**Solution:** Draw right triangle: opp=3, adj=4, hyp=5.  $\cos(\arctan(3/4)) = 4/5$

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## Unit 5 Solving Trigonometric Equations

### Strategy

1. Isolate the trig function
2. Find reference angle using inverse trig
3. Use CAST / quadrant rules to find all solutions in  $[0, 2\pi)$
4. Add period multiples ( $+ 2n\pi$  for sin/cos,  $+ n\pi$  for tan) if general solution required
5. Check domain restrictions

### Worked Example

Example: Solve  $2\sin^2x - \sin x - 1 = 0$  on  $[0, 2\pi)$ .

**Solution:** Factor:  $(2\sin x + 1)(\sin x - 1) = 0 \rightarrow \sin x = -1/2$  or  $1 \rightarrow x = 7\pi/6, 11\pi/6, \pi/2$

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## Unit 6 Law of Sines & Cosines

### Laws

Law of Sines:  $a/\sin A = b/\sin B = c/\sin C$

Law of Cosines:  $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

$\cos A = (b^2 + c^2 - a^2)/(2bc)$

Area of triangle:  $K = (1/2)ab \sin C$

Heron's formula:  $K = \sqrt{s(s-a)(s-b)(s-c)}$ ,  $s = (a+b+c)/2$

### Worked Example

Example: In triangle ABC,  $a=7$ ,  $b=5$ ,  $C=60^\circ$ . Find  $c$ .

**Solution:**  $c^2 = 49 + 25 - 70\cos 60^\circ = 74 - 35 = 39 \rightarrow c = \sqrt{39} \approx 6.24$

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## Unit 7 Polar Coordinates & Trigonometric Form of Complex Numbers

### Key Conversions

Polar  $\rightarrow$  Rect:  $x = r \cos \theta$ ,  $y = r \sin \theta$

Rect  $\rightarrow$  Polar:  $r = \sqrt{x^2 + y^2}$ ,  $\theta = \arctan(y/x)$  (adjust quadrant!)

Complex trig form:  $z = r(\cos\theta + i \sin\theta)$  where  $r=|z|$ ,  $\theta=\arg(z)$

DeMoivre's Theorem:  $z^n = r^n (\cos(n\theta) + i \sin(n\theta))$

$n$ th roots:  $z_k = r^{1/n}[\cos((\theta+2k\pi)/n) + i \sin((\theta+2k\pi)/n)]$ ,  $k=0, 1, \dots, n-1$

### Worked Example

Example: Write  $z = 1 + i$  in trig form, then find  $z^4$ .

**Solution:**  $r=\sqrt{2}$ ,  $\theta=\pi/4$ .  $z^4 = (\sqrt{2})^4(\cos \pi + i \sin \pi) = 4(-1+0i) = -4$

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## PRACTICE PROBLEMS

Show all work. Answers are collected at the end.

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— Unit 1 —

### Problem 1

A circle has radius 6 cm. A central angle subtends an arc of 10 cm.

- (a) Find the central angle in radians.
- (b) Find the area of the sector.

Work space:

### Problem 2

A gear rotates at 120 revolutions per minute.

- (a) Find the angular speed  $\omega$  in radians per second.
- (b) If the gear radius is 5 cm, find the linear speed of a point on its rim.

Work space:



### Problem 5

The graph of  $y = A \sin(Bx)$  passes through  $(\pi/4, 3)$  and has period  $\pi$ .  
Find  $A$  and  $B$ , then write the equation.

Work space:

— Unit 3 —

### Problem 6

Prove the identity:

$$(\tan x + \cot x)^2 - (\tan x - \cot x)^2 = 4$$

Work space:



### Problem 9

Prove:

$$\cos(3x) = 4\cos^3x - 3\cos x$$

Work space:


### Problem 10

Verify the identity:

$$\sin(4x) / (1 + \cos(4x)) = \tan(2x)$$

Work space:


### Problem 11

Evaluate without a calculator:

$$\sin(\arccos(-\sqrt{3}/2) + \arctan(1))$$

Work space:


### Problem 12

Simplify the expression:

$$\tan(\arcsin(x / \sqrt{x^2+4})) \text{ for } x > 0$$

Work space:






**Problem 17**

A triangular plot has sides 9 m, 12 m, and 15 m.

- (a) Show it is a right triangle.
- (b) Find the area using the formula  $K = (1/2)ab \sin C$ .

Work space:

— Unit 7 —

**Problem 18**

Convert the polar equation  $r = 6\sin(\theta)$  to rectangular form and identify the curve.

Work space:



## ANSWER KEY & SOLUTIONS

### Problem 1 [Unit 1]

A circle has radius 6 cm. A central angle subtends an arc of 10 cm. (a) Find the central angle in radians. (b) Find the area of the sector.

$$(a) s = r \cdot \theta \rightarrow \theta = s/r = 10/6 = 5/3 \text{ rad}$$

$$(b) A = (1/2)r^2 \cdot \theta = (1/2)(36)(5/3) = 30 \text{ cm}^2 \dots \text{re-check:}$$

$$(1/2) \cdot 36 \cdot (5/3) = 18 \cdot (5/3) = 30 \text{ cm}^2$$

Note: short answer corrected to 30 cm<sup>2</sup>.

### Problem 2 [Unit 1]

A gear rotates at 120 revolutions per minute. (a) Find the angular speed  $\omega$  in radians per second. (b) If the gear radius is 5 cm, find the linear speed of a point on its rim.

$$(a) 120 \text{ rev/min} = 120 \cdot (2\pi)/60 \text{ rad/s} = 4\pi \text{ rad/s}$$

$$(b) v = r \cdot \omega = 5 \cdot 4\pi = 20\pi \text{ cm/s}$$

### Problem 3 [Unit 1]

Without a calculator, find the exact value of:  $\sin(-5\pi/3) + \cos(11\pi/6) - \tan(5\pi/4)$

$$\sin(-5\pi/3) = -\sin(5\pi/3) = -(-\sin(\pi/3)) = \sin(\pi/3) = \sqrt{3}/2$$

$$\cos(11\pi/6) = \cos(2\pi - \pi/6) = \cos(\pi/6) = \sqrt{3}/2$$

$$\tan(5\pi/4) = \tan(\pi + \pi/4) = \tan(\pi/4) = 1$$

$$\text{Sum} = \sqrt{3}/2 + \sqrt{3}/2 - 1 = \sqrt{3} - 1$$

### Problem 4 [Unit 2]

For the function  $y = -2 \cos(\pi x/3 + \pi/6) + 1$ : (a) State the amplitude, period, phase shift, and vertical shift. (b) Find all  $x$ -values on  $[0, 6]$  where the function reaches its maximum.

$$\text{Rewrite: } y = -2\cos((\pi/3)(x + 1/2)) + 1$$

$$(a) \text{ Amplitude} = |-2| = 2; \text{ Period} = 2\pi/(\pi/3) = 6$$

$$\text{Phase shift} = -1/2 \text{ (i.e. } 1/2 \text{ unit to the LEFT); Vertical shift } D=1$$

$$(b) \text{ Maximum of } -2\cos(\dots)+1 \text{ occurs when } \cos(\dots) = -1$$

$$(\pi/3)(x+1/2) = \pi \rightarrow x+1/2 = 3 \rightarrow x = 5/2 \checkmark \text{ (in } [0,6])$$

### Problem 5 [Unit 2]

The graph of  $y = A \sin(Bx)$  passes through  $(\pi/4, 3)$  and has period  $\pi$ . Find  $A$  and  $B$ , then write the equation.

$$\text{Period} = 2\pi/B = \pi \rightarrow B = 2$$

$$\text{Substitute } (\pi/4, 3): 3 = A \sin(2 \cdot \pi/4) = A \sin(\pi/2) = A$$

$$\text{Therefore } A = 3 \text{ and } y = 3 \sin(2x)$$

### Problem 6 [Unit 3]

Prove the identity:  $(\tan x + \cot x)^2 - (\tan x - \cot x)^2 = 4$

$$\text{LHS} = (\tan x + \cot x)^2 - (\tan x - \cot x)^2$$

Use  $(a+b)^2 - (a-b)^2 = 4ab$ :

$$= 4 \cdot \tan x \cdot \cot x = 4 \cdot (\sin x / \cos x) \cdot (\cos x / \sin x) = 4 \cdot 1 = 4 \checkmark$$

### Problem 7 [Unit 3]

Given that  $\sin A = 3/5$  (A in Quadrant I) and  $\cos B = -5/13$  (B in Quadrant II), find the exact value of  $\sin(A - B)$ .

$$\sin A = 3/5 \rightarrow \cos A = 4/5 \text{ (Q I)}$$

$$\cos B = -5/13 \rightarrow \sin B = 12/13 \text{ (Q II, sin positive)}$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$= (3/5)(-5/13) - (4/5)(12/13)$$

$$= -15/65 - 48/65 = -63/65$$

Re-check sign: A in Q1, B in Q2 with  $\cos B = -5/13$ ,  $\sin B = +12/13$  ✓

$$\sin(A-B) = -15/65 - 48/65 = -63/65$$

Note: answer is  $-63/65$  (not  $+63/65$ ; original short answer corrected).

### Problem 8 [Unit 3]

Use the double-angle identity to simplify:  $f(x) = (\sin x + \cos x)^2 - 1$  and find all  $x$  in  $[0, 2\pi]$  where  $f(x) = 0$ .

$$(\sin x + \cos x)^2 - 1 = \sin^2 x + 2\sin x \cos x + \cos^2 x - 1$$

$$= 1 + 2\sin x \cos x - 1 = 2\sin x \cos x = \sin(2x)$$

$$\sin(2x) = 0 \rightarrow 2x = 0, \pi, 2\pi, 3\pi \rightarrow x = 0, \pi/2, \pi, 3\pi/2$$

### Problem 9 [Unit 3]

Prove:  $\cos(3x) = 4\cos^3 x - 3\cos x$

$$\cos(3x) = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$$

$$= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x$$

$$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$$

$$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$$

$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$

$$= 4\cos^3 x - 3\cos x \checkmark$$

### Problem 10 [Unit 3]

Verify the identity:  $\sin(4x) / (1 + \cos(4x)) = \tan(2x)$

Use half-angle form: for any angle  $u$ ,  $\sin u / (1 + \cos u) = \tan(u/2)$

$$\text{Set } u = 4x: \sin(4x) / (1 + \cos(4x)) = \tan(4x/2) = \tan(2x) \checkmark$$

OR directly:  $\sin(4x)=2\sin 2x \cos 2x$ ,  $1+\cos(4x)=2\cos^2(2x)$

Ratio =  $2\sin 2x \cos 2x / (2\cos^2 2x) = \sin 2x / \cos 2x = \tan 2x \checkmark$

### Problem 11 [Unit 4]

Evaluate without a calculator:  $\sin(\arccos(-\sqrt{3}/2) + \arctan(1))$

$\arccos(-\sqrt{3}/2) = 5\pi/6$  (range of arccos is  $[0, \pi]$ )

$\arctan(1) = \pi/4$

$\sin(5\pi/6 + \pi/4) = \sin 5\pi/6 \cos \pi/4 + \cos 5\pi/6 \sin \pi/4$

$= (1/2)(\sqrt{2}/2) + (-\sqrt{3}/2)(\sqrt{2}/2)$

$= \sqrt{2}/4 - \sqrt{6}/4 = (\sqrt{2} - \sqrt{6})/4$

### Problem 12 [Unit 4]

Simplify the expression:  $\tan(\arcsin(x / \sqrt{x^2+4}))$  for  $x > 0$

Let  $\theta = \arcsin(x/\sqrt{x^2+4})$

Draw right triangle: opposite =  $x$ , hypotenuse =  $\sqrt{x^2+4}$

adjacent =  $\sqrt{((x^2+4) - x^2)} = \sqrt{4} = 2$

$\tan(\theta) = \text{opposite/adjacent} = x/2$

### Problem 13 [Unit 5]

Solve on  $[0, 2\pi)$ :  $2\cos^2 x + 3\sin x - 3 = 0$

Replace  $\cos^2 x = 1 - \sin^2 x$ :

$2(1 - \sin^2 x) + 3\sin x - 3 = 0$

$-2\sin^2 x + 3\sin x - 1 = 0 \rightarrow 2\sin^2 x - 3\sin x + 1 = 0$

$(2\sin x - 1)(\sin x - 1) = 0 \rightarrow \sin x = 1/2$  or  $\sin x = 1$

$\sin x = 1/2 \rightarrow x = \pi/6, 5\pi/6$ ;  $\sin x = 1 \rightarrow x = \pi/2$

### Problem 14 [Unit 5]

Find all solutions in  $[0, 2\pi)$  of:  $\sin(2x) = \cos x$

$\sin(2x) = 2\sin x \cos x$ , so:  $2\sin x \cos x = \cos x$

$2\sin x \cos x - \cos x = 0 \rightarrow \cos x(2\sin x - 1) = 0$

$\cos x = 0 \rightarrow x = \pi/2, 3\pi/2$

$\sin x = 1/2 \rightarrow x = \pi/6, 5\pi/6$

All four solutions are valid.

### Problem 15 [Unit 5]

Solve the equation for exact general solutions:  $\tan^2 x - (\sqrt{3} + 1)\tan x + \sqrt{3} = 0$

Factor:  $(\tan x - \sqrt{3})(\tan x - 1) = 0$

$$\tan x = \sqrt{3} \rightarrow x = \pi/3 + n \cdot \pi$$

$$\tan x = 1 \rightarrow x = \pi/4 + n \cdot \pi$$

### Problem 16 [Unit 6]

In triangle ABC:  $a = 8$ ,  $b = 5$ , angle  $A = 30^\circ$ . Determine the number of possible triangles and find all solutions for angle B.

$$\text{Use Law of Sines: } \sin B/b = \sin A/a \rightarrow \sin B = 5 \cdot \sin 30^\circ / 8 = 5/16$$

$$\sin B = 5/16 \approx 0.3125 \rightarrow B_1 \approx 18.21^\circ \text{ and } B_2 = 180^\circ - 18.21^\circ = 161.79^\circ$$

$$\text{Check } B_2: A+B_2 = 30^\circ + 161.79^\circ = 191.79^\circ > 180^\circ \rightarrow B_2 \text{ is INVALID}$$

$$\text{Only ONE triangle exists: } B \approx 18.21^\circ$$

Correction: since  $A+B_2 > 180^\circ$ , only one triangle. Short answer updated.

### Problem 17 [Unit 6]

A triangular plot has sides 9 m, 12 m, and 15 m. (a) Show it is a right triangle. (b) Find the area using the formula  $K = (1/2)ab \sin C$ .

$$(a) 81 + 144 = 225 = 15^2 \rightarrow \text{right triangle with hypotenuse 15 m}$$

$$(b) \text{ The right angle } C \text{ is opposite the hypotenuse, so } C=90^\circ$$

$$K = (1/2)(9)(12)\sin 90^\circ = (1/2)(108)(1) = 54 \text{ m}^2$$

### Problem 18 [Unit 7]

Convert the polar equation  $r = 6\sin(\theta)$  to rectangular form and identify the curve.

$$\text{Multiply both sides by } r: r^2 = 6r \cdot \sin(\theta)$$

$$x^2 + y^2 = 6y \rightarrow x^2 + y^2 - 6y = 0$$

$$\text{Complete the square: } x^2 + (y-3)^2 = 9 \checkmark$$

### Problem 19 [Unit 7]

Use DeMoivre's Theorem to find  $(1 - \sqrt{3}i)^6$  in rectangular form.

$$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\theta = \arctan(-\sqrt{3}/1) = -\pi/3 \text{ (Quadrant IV)}$$

$$(1 - \sqrt{3}i)^6 = 2^6 [\cos(6 \cdot (-\pi/3)) + i \sin(-2\pi)]$$

$$= 64 [\cos(-2\pi) + i \sin(-2\pi)] = 64[1 + 0i] = 64$$

### Problem 20 [Unit 7]

Find all cube roots of  $z = 27i$  in trigonometric form, then convert each to  $a + bi$ .

$$z = 27i = 27(\cos(\pi/2) + i \sin(\pi/2))$$

$$\text{Cube roots: } r = 27^{1/3} = 3, \text{ angles} = (\pi/2 + 2k\pi)/3 \text{ for } k=0,1,2$$

$$k=0: \theta = \pi/6 \rightarrow 3(\cos \pi/6 + i \sin \pi/6) = 3(\sqrt{3}/2 + i/2) = 3\sqrt{3}/2 + 3i/2$$

$$k=1: \theta = 5\pi/6 \rightarrow 3(\cos 5\pi/6 + i \sin 5\pi/6) = 3(-\sqrt{3}/2 + i/2) = -3\sqrt{3}/2 + 3i/2$$

$$k=2: \theta=3\pi/2 \rightarrow 3(\cos 3\pi/2 + i \sin 3\pi/2) = 3(0 - i) = -3i$$

Three roots:  $3\sqrt{3}/2 + 3i/2$ ,  $-3\sqrt{3}/2 + 3i/2$ ,  $-3i$