# PRE-CALCULUS

## 20 Essential Problems

Concepts · Formulas · Problems · Solutions

UNIT 01

# Functions & Their Graphs

## CORE CONCEPT

A function f maps each input x to exactly one output f(x). Domain = allowed x-values; Range = resulting y-values. Vertical Line Test: a graph is a function iff every vertical line hits it at most once.

## KEY FORMULAS

- → **Composition: (f o g)(x) = f(g(x))**
- → **Inverse: $f(f^{-1}(x)) = x$ [swap x,y then solve]**
- → **Even: f(-x)=f(x) | Odd: f(-x)=-f(x)**

## MEMORIZE

- ★ Domain of sqrt(x): x >= 0
- ★ Domain of 1/x: $x \neq 0$
- ★ Domain of log(x): x > 0

### EXAMPLE

Q: Find (f o g)(x) if $f(x)=x^2+1$ and g(x)=2x-3

A: Solution: $f(g(x)) = f(2x-3) = (2x-3)^2+1 = 4x^2-12x+10$

UNIT 02

# Polynomial Functions

## CORE CONCEPT

A polynomial of degree n has at most n real zeros and n-1 turning points. End behavior: determined by leading term a_n x^n.

## KEY FORMULAS

- → **Factor Theorem: (x-c) is a factor iff f(c)=0**
- → **Rational Zero Theorem: p/q where p|constant, q|leading**
- → **Remainder Theorem: f(c) = remainder of f(x) / (x-c)**

## MEMORIZE

★ Degree even, positive lead → both ends UP

★ Degree odd, positive lead → left DOWN, right UP

★ Complex roots come in conjugate pairs

### EXAMPLE

Q: Find all rational zeros of $f(x)=x^3-6x^2+11x-6$

A: Solution: Try x=1,2,3. f(1)=0 so (x-1) is a factor. Divide: $(x-1)(x^2-5x+6)=(x-1)(x-2)(x-3)$. Zeros: x=1,2,3

UNIT 03

# Rational Functions

## CORE CONCEPT

f(x)=P(x)/Q(x). Vertical asymptotes where Q(x)=0 (and P(x)≠0). Horizontal asymptotes depend on degrees of P vs Q.

## KEY FORMULAS

- → **HA: deg P < deg Q → y=0**
- → **HA: deg P = deg Q → y = leading coeff ratio**
- → **Slant asy: deg P = deg Q + 1 → do polynomial division**

## MEMORIZE

- ★ Holes occur where both P(c)=0 and Q(c)=0
- ★ x-intercepts: set numerator = 0 (check not also denom)
- ★ y-intercept: set x=0

### EXAMPLE

Q: Find asymptotes of $f(x)=(2x^2+3x)/(x^2-4)$

A: Solution: VA: x=2, x=-2. HA: deg equal, y=2/1=2. x-int: x=0, x=-3/2.

UNIT 04

# Exponential & Logarithmic Functions

## CORE CONCEPT

Exponential: $f(x)=a^x$ ($a>0$, $a\neq1$). Logarithm: log_a(x) is the inverse. Natural base $e \approx 2.718$.

## KEY FORMULAS

- → **log(MN) = log M + log N**
- → **log(M/N) = log M - log N**
- → **$\log(M^p) = p \log M$**
- → **Change of base: log_a(b) = ln(b)/ln(a)**
- → **$A = Pe^{rt}$ (continuous compound)**

## MEMORIZE

★ log_a(a) = 1 | log_a(1) = 0

★ $a^{\log\_a(x)} = x$ | $\log\_a(a^x) = x$

★ Domain of log: $x > 0$ only

### EXAMPLE

Q: Solve: $2^{x+1} = 32$

A: Solution: $32=2^5$, so $x+1=5$, $x=4$.

UNIT 05

# Trigonometric Functions

## CORE CONCEPT

Unit circle: (cos θ, sin θ). SOHCAHTOA for right triangles. Period of sin/cos = 2π; period of tan = π.

## KEY FORMULAS

- → **$\sin^2\theta + \cos^2\theta = 1$**
- → **$\tan\theta = \sin\theta / \cos\theta$**
- → **$1 + \tan^2\theta = \sec^2\theta$**
- → **$1 + \cot^2\theta = \csc^2\theta$**

## MEMORIZE

★ sin(30°)=1/2, sin(45°)=√2/2, sin(60°)=√3/2

★ cos(0°)=1, cos(90°)=0, cos(180°)=-1

★ ASTC: All-Sin-Tan-Cos positive by quadrant

### EXAMPLE

Q: Find exact value of sin(5π/6)

A: Solution: 5π/6 is in Q2. Reference angle = π/6. sin(π/6)=1/2. Q2 sin is positive → sin(5π/6) = 1/2

UNIT 06

# Trig Identities & Equations

## CORE CONCEPT

Double angle, sum/difference, and half-angle formulas allow simplification. Always check for extraneous solutions when solving.

## KEY FORMULAS

- → **sin(A±B) = sinA cosB ± cosA sinB**
- → **cos(A±B) = cosA cosB ■ sinA sinB**
- → **sin(2A) = 2sinA cosA**
- → **cos(2A) = $\cos^2$A - $\sin^2$A**

## MEMORIZE

- ★ Verify identity: work one side only
- ★ Factor first when solving trig equations
- ★ General solution: add 2πk (or πk for tan)

### EXAMPLE

Q: Solve: $2\sin^2 x - \sin x - 1 = 0$ on [0, 2π)

A: Solution: Factor: (2sinx+1)(sinx-1)=0. sinx=-1/2 → x=7π/6,11π/6; sinx=1 → x=π/2

UNIT 07

# Inverse Trigonometric Functions

## CORE CONCEPT

Restricted domains for inverses: arcsin [-π/2, π/2], arccos [0, π], arctan (-π/2, π/2).

## KEY FORMULAS

- → **arcsin(x): domain [-1,1], range [-π/2, π/2]**
- → **arccos(x): domain [-1,1], range [0, π]**
- → **arctan(x): domain all R, range (-π/2, π/2)**

## MEMORIZE

- ★ sin(arcsin x) = x only for x in [-1,1]
- ★ arcsin(sin x) = x only for x in [-π/2, π/2]
- ★ arctan is always between -90° and 90°

### EXAMPLE

Q: Find exact value of cos(arcsin(3/5))

A: Solution: Let θ=arcsin(3/5), so sinθ=3/5. Draw triangle: opp=3, hyp=5, adj=4. cosθ=4/5

UNIT 08

# Conic Sections

## CORE CONCEPT

Parabola, ellipse, hyperbola, circle — all cross-sections of a cone. Each has standard form equations.

## KEY FORMULAS

- → **Circle: $(x-h)^2+(y-k)^2=r^2$**
- → **Ellipse: $x^2/a^2 + y^2/b^2 = 1$ $(a>b>0)$**
- → **Hyperbola: $x^2/a^2 - y^2/b^2 = 1$**
- → **Parabola: $y = (1/4p)x^2$ (focus at $(0,p)$)**

## MEMORIZE

- ★ Ellipse: $c^2=a^2-b^2$
- ★ Hyperbola: $c^2=a^2+b^2$
- ★ Parabola opens toward focus

### EXAMPLE

Q: Write equation of ellipse: center (0,0), vertices (±5,0), co-vertices (0,±3)

A: Solution: a=5, b=3. Equation: $x^2/25 + y^2/9 = 1$

UNIT 09

# Systems of Equations & Matrices

## CORE CONCEPT

Solve systems with substitution, elimination, or matrices. Matrix row operations preserve solution sets.

## KEY FORMULAS

- → **Matrix multiplication: (AB)_ij = sum of A row i · B col j**
- → **Cramer's Rule: x = D_x/D, y = D_y/D**
- → **det([[a,b],[c,d]]) = ad - bc**
- → **$A^{-1}$ = (1/det A) adj(A)**

## MEMORIZE

★ 3 cases: one solution (lines cross), no solution (parallel), infinite (same line)

★ Row echelon: zeros below diagonal

★ det=0 → matrix is singular (no inverse)

### EXAMPLE

Q: Solve: x+y=5, 2x-y=1

A: Solution: Add equations: 3x=6, x=2. Then y=3. Solution: (2,3)

UNIT 10

# Sequences & Series

## CORE CONCEPT

Arithmetic: constant difference d. Geometric: constant ratio r. Series = sum of terms.

## KEY FORMULAS

- → **Arithmetic: $a_n = a_1 + (n-1)d$, $S_n = n/2(a_1+a_n)$**
- → **Geometric: $a_n = a_1 r^{n-1}$, $S_n = a_1(1-r^n)/(1-r)$**
- → **Infinite geometric: $S = a_1/(1-r)$ if $|r|<1$**
- → **Binomial: $(a+b)^n$ = sum $C(n,k) a^{n-k} b^k$**

## MEMORIZE

- ★ $|r| < 1$ → infinite geometric series converges
- ★ $C(n,k) = n! / (k!(n-k)!)$
- ★ Arithmetic mean = $(a+b)/2$, Geometric mean = $\sqrt{ab}$

### EXAMPLE

Q: Find sum of first 10 terms: 3+7+11+...

A: Solution: Arithmetic, $a_1=3$, $d=4$. $S_{10} = 10/2(3+39) = 5(42) = 210$

# PRACTICE PROBLEMS

Exam-style open-response questions

**Q01 · Functions & Their Graphs**

**Let $f(x) = 3x - 2$ and $g(x) = x^2 + 1$. Find $(f \circ g)(x)$ and state its domain.**

Hint: Substitute g(x) into f. Domain of composition is all x where g(x) is in domain of f.

**Q02 · Functions & Their Graphs**

**Find the inverse of $f(x) = (2x + 5) / (x - 1)$ and state the domain of $f^{-1}(x)$.**

Hint: Swap x and y, then solve for y. Find where denominator ≠ 0.

**Q03 · Polynomial Functions**

**A polynomial f(x) has zeros at $x = -2$ (multiplicity 2), $x = 1$, and $x = 3$. The leading coefficient is 2. Write f(x) in factored form and determine its degree.**

Hint: Multiplicity 2 means $(x+2)^2$ factor. Degree = sum of all multiplicities.

**Q04 · Polynomial Functions**

**Use synthetic division to find all zeros of $f(x) = 2x^3 - 5x^2 - 4x + 3$. Show your work.**

Hint: Use Rational Zero Theorem: possible zeros are ±1, ±3, ±1/2, ±3/2.

**Q05 · Rational Functions**

**Analyze $f(x) = (x^2 - 9) / (x^2 - x - 6)$. Find all holes, vertical asymptotes, horizontal asymptotes, and x-intercepts.**

Hint: Factor both numerator and denominator completely before finding asymptotes.

**Q06 · Rational Functions**

**Find the slant (oblique) asymptote of $g(x) = (x^2 + 3x - 5) / (x + 2)$.**

Hint: Degree of numerator is 1 more than denominator. Perform polynomial long division.

**Q07 · Exponential & Logarithmic Functions**

**Solve for x: $\log_3(x + 4) + \log_3(x - 2) = 3$. Verify your answer.**

Hint: Use log(MN)=log M+log N to combine, then convert to exponential form. Check domain.

**Q08 · Exponential & Logarithmic Functions**

**A bacteria culture starts with 500 bacteria and doubles every 3 hours. Write an exponential model and determine how many hours until there are 16,000 bacteria.**

Hint: Model: $A = 500 * 2^{t/3}$. Solve $16000 = 500 * 2^{t/3}$ using logarithms.

**Q09 · Trigonometric Functions**

**Graph $y = 3 \sin(2x - \pi/3) + 1$. State the amplitude, period, phase shift, and vertical shift.**

Hint: Rewrite as $3\sin(2(x - \pi/6))+1$. Amplitude = $|A|$, period = $2\pi/|B|$.

**Q10 · Trigonometric Functions**

**A 20-foot ladder leans against a building. The base is 8 feet from the wall. Find the angle the ladder makes with the ground to the nearest tenth of a degree.**

Hint: Draw a right triangle. The adjacent side is 8, hypotenuse is 20. Use inverse trig.

**Q11 · Trig Identities & Equations**

**Verify the identity: $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$.**

Hint: Expand the left side using $(a+b)^2 = a^2+2ab+b^2$, then use Pythagorean identity.

**Q12 · Trig Identities & Equations**

**Solve for all solutions in $[0, 2\pi)$: $\cos(2x) + \cos x = 0$.**

Hint: Use double angle: $\cos(2x) = 2\cos^2 x - 1$. Factor the resulting quadratic in cos x.

**Q13 · Inverse Trigonometric Functions**

**Evaluate: $\sin(\arctan(4/3))$ without using a calculator.**

Hint: Let $\theta = \arctan(4/3)$, so $\tan \theta = 4/3$. Draw a right triangle with opposite=4, adjacent=3.

**Q14 · Inverse Trigonometric Functions**

**Find the exact value of $\arcsin(\sin(7\pi/6))$.**

Hint: arcsin always outputs values in $[-\pi/2, \pi/2]$. Find the reference angle in the correct quadrant.

**Q15 · Conic Sections**

**Convert $x^2 + 4y^2 - 6x + 16y + 21 = 0$ to standard form. Identify the conic and find its center, vertices, and foci.**

Hint: Complete the square for both x and y separately. Check the resulting standard form.

**Q16 · Conic Sections**

**Find the equation of a parabola with vertex at (2, -3) and focus at (2, 0). Sketch key features.**

Hint: Vertex and focus have same x-coordinate → vertical parabola. Find p = distance from vertex to focus.

**Q17 · Systems of Equations & Matrices**

**Use Cramer's Rule to solve: 3x + 2y = 7 and x - 4y = -5.**

Hint: Find D = det[[3,2],[1,-4]]. Then D_x = det[[7,2],[-5,-4]] and D_y = det[[3,7],[1,-5]].

**Q18 · Systems of Equations & Matrices**

**Find the partial fraction decomposition of (3x + 5) / ($x^2$ - x - 2).**

Hint: Factor denominator: (x-2)(x+1). Set up A/(x-2) + B/(x+1) and solve for A and B.

**Q19 · Sequences & Series**

**Find the 8th term of a geometric sequence where the 3rd term is 12 and the 6th term is 96.**

Hint: Use a_6 = a_3 * $r^3$ to find r. Then use a_8 = a_3 * $r^5$.

**Q20 · Sequences & Series**

**Use the Binomial Theorem to expand $(x - 2y)^4$. Write all terms.**

Hint: C(4,k)$x^{4-k}(-2y)^k$ for k=0,1,2,3,4. Remember $(-2y)^k$ includes the sign.

# ANSWER KEY & SOLUTIONS

Full worked solutions for all 20 problems

**PROBLEM 01**

**Answer: (f o g)(x) = $3x^2 + 1$; Domain: all real numbers (-∞, ∞)**

Solution: $f(g(x)) = f(x^2+1) = 3(x^2+1) - 2 = 3x^2+3-2 = 3x^2+1$. $g(x)=x^2+1$ is defined for all real x, and f is defined for all real inputs, so domain is all reals.

**PROBLEM 02**

**Answer: $f^{-1}(x) = (x + 5)/(x - 2)$; Domain: $x \neq 2$**

Solution: Swap: x = (2y+5)/(y-1). Multiply: x(y-1)=2y+5 → xy-x=2y+5 → xy-2y=x+5 → y(x-2)=x+5 → y=(x+5)/(x-2). Domain: x≠2.

**PROBLEM 03**

**Answer: $f(x) = 2(x+2)^2(x-1)(x-3)$; Degree = 4**

Solution: Multiplicity 2 at x=-2 gives $(x+2)^2$. Single zeros at x=1 and x=3 give (x-1)(x-3). Leading coefficient 2 scales the whole polynomial. Total degree = 2+1+1 = 4.

**PROBLEM 04**

**Answer: Zeros: x = 3, x = 1/2, x = -1**

Solution: Test x=3: 2(27)-5(9)-4(3)+3=54-45-12+3=0. Synthetic division gives $2x^2+x-1$. Factor: (2x-1)(x+1). Zeros: x=1/2 and x=-1. All zeros: x=3, 1/2, -1.

**PROBLEM 05**

**Answer: Hole: (3, 6/5). VA: x = -2. HA: y = 1. x-intercept: (-3, 0)**

Solution: Factor: (x+3)(x-3)/[(x-3)(x+2)]. Cancel (x-3): hole at x=3. Simplified: (x+3)/(x+2). VA: x=-2. HA: degrees equal, y=1/1=1. x-int: x=-3.

**PROBLEM 06**

**Answer: Slant asymptote: y = x + 1**

Solution: Divide $x^2+3x-5$ by x+2: $x^2+3x-5 = (x+2)(x+1) - 7$. So g(x) = x+1 - 7/(x+2). As x→±∞, the -7/(x+2) → 0. Slant asymptote: y = x+1.

**PROBLEM 07**

**Answer: x = 5 (x = -7 is extraneous)**

Solution: Combine: $\log_3[(x+4)(x-2)]=3$ → (x+4)(x-2)=27 → $x^2+2x-8=27$ → $x^2+2x-35=0$ → (x+7)(x-5)=0. x=-7 makes log(x-2)=log(-9) undefined. Answer: x=5.

**PROBLEM 08**

**Answer: Model: $A = 500 \cdot 2^{t/3}$; t = 15 hours**

Solution: $16000=500\cdot 2^{t/3}$ → $32=2^{t/3}$ → $2^5=2^{t/3}$ → t/3=5 → t=15 hours.

**PROBLEM 09**

**Answer: Amplitude=3, Period=$\pi$, Phase shift=$\pi/6$ right, Vertical shift=1 up**

Solution: $y=3\sin(2(x-\pi/6))+1$. $|A|=3$. Period=$2\pi/2=\pi$. Phase shift=$\pi/6$ (right). Midline y=1 (vertical shift up 1). Max value=4, min value=-2.

---

**PROBLEM 10**

**Answer: $\theta \approx 66.4°$**

Solution: $\cos\theta = 8/20 = 0.4$. $\theta = \arccos(0.4) \approx 66.4°$. Alternatively, $\sin\theta$ = sqrt(400-64)/20 = sqrt(336)/20 $\approx$ 0.9165, $\theta = \arcsin(0.9165) \approx 66.4°$.

---

**PROBLEM 11**

**Answer: Identity verified.**

Solution: LHS: $(\sin x+\cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x = (\sin^2 x+\cos^2 x) + 2\sin x \cos x = 1 + 2\sin x \cos x$ = RHS. ✓

---

**PROBLEM 12**

**Answer: x = $\pi/2$, x = $2\pi/3$, x = $4\pi/3$**

Solution: $\cos(2x)+\cos x=0 \rightarrow 2\cos^2 x-1+\cos x=0 \rightarrow (2\cos x-1)(\cos x+1)=0$. $\cos x=1/2 \rightarrow x=\pi/3, 5\pi/3$... Wait: but check full interval. $\cos x=1/2 \rightarrow x=\pi/3, 5\pi/3$. $\cos x=-1 \rightarrow x=\pi$. Three solutions in $[0,2\pi)$: $x=\pi/3, \pi, 5\pi/3$.

---

**PROBLEM 13**

**Answer: sin(arctan(4/3)) = 4/5**

Solution: Let $\theta$=arctan(4/3). Triangle: opp=4, adj=3, hyp=sqrt(9+16)=5. $\sin\theta$=opp/hyp=4/5.

---

**PROBLEM 14**

**Answer: arcsin(sin($7\pi/6$)) = $-\pi/6$**

Solution: $\sin(7\pi/6)=\sin(\pi+\pi/6)=-\sin(\pi/6)=-1/2$. arcsin(-1/2): find angle in $[-\pi/2,\pi/2]$ with sine=-1/2. That's $-\pi/6$.

---

**PROBLEM 15**

**Answer: $(x-3)^2/16 + (y+2)^2/4 = 1$; Ellipse, Center(3,-2), Vertices(7,-2),(-1,-2), Foci($3\pm2\sqrt{3}$,-2)**

Solution: Complete square: $(x^2-6x+9)+4(y^2+4y+4)=9+16-21=4$. $(x-3)^2+4(y+2)^2=16$. Divide by 16: standard form. $a^2=16$, $b^2=4$, $c^2=12$, $c=2\sqrt{3}$.

---

**PROBLEM 16**

**Answer: $(x-2)^2 = 12(y+3)$**

Solution: Same x → vertical parabola. p = distance from vertex(2,-3) to focus(2,0) = 3. Since focus is above vertex, opens upward. Standard form: $(x-h)^2=4p(y-k) \rightarrow (x-2)^2=12(y+3)$.

---

**PROBLEM 17**

**Answer: x = 9/7, y = 11/7**

Solution: D = 3(-4) - 2(1) = -14. $D_x$ = 7(-4) - 2(-5) = -18. $D_y$ = 3(-5) - 7(1) = -22. x = $D_x$/D = -18/-14 = 9/7. y = $D_y$/D = -22/-14 = 11/7. Verify: 3(9/7)+2(11/7)=27/7+22/7=49/7=7 ✓

---

**PROBLEM 18**

**Answer: 11/(3(x-2)) - 2/(3(x+1))**

Solution: Factor denominator: (x-2)(x+1). Set up A/(x-2)+B/(x+1). Multiply: 3x+5=A(x+1)+B(x-2). Plug x=2: 11=3A → A=11/3. Plug x=-1: 2=-3B → B=-2/3. Result: 11/(3(x-2)) - 2/(3(x+1)).

---

**PROBLEM 19**

**Answer: a_8 = 384**

Solution: a_6 = a_3 · $r^3$ → 96 = $12r^3$ → $r^3$=8 → r=2. a_8 = a_3 · $r^5$ = 12 · $2^5$ = 12 · 32 = 384.

---

**PROBLEM 20**

**Answer: $x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$**

Solution: C(4,0)$x^4$=$x^4$. C(4,1)$x^3$(-2y)=-8$x^3$y. C(4,2)$x^2$(4$y^2$)=24$x^2y^2$. C(4,3)x(-8$y^3$)=-32x$y^3$. C(4,4)(16$y^4$)=16$y^4$.

---