PRE-CALCULUS MASTER GUIDE

Complete Concept Review · 20 Exam-Style Problems · Full Solutions

Covering All Core Topics:

Functions · Polynomials · Exponential & Logarithmic · Trigonometry

Sequences & Series · Conic Sections · Limits · Vectors

SAT / AP Calculus / IB Math Style · Subjective Format

UNIT 01

Functions & Their Properties

Domain, Range, Composition, Inverses

Core Concepts

A function f maps each input x to exactly one output f(x). The domain is the set of valid inputs; the range is the set of outputs.

Vertical Line Test: a graph is a function if every vertical line hits it at most once.

Even function: f(-x) = f(x) (symmetric about y-axis). Odd function: f(-x) = -f(x) (symmetric about origin).

Composition: (f o g)(x) = f(g(x)). Note: order matters — f o g is generally not equal to g o f.

Inverse f^(-1) exists iff f is one-to-one (passes Horizontal Line Test). If (a, b) is on f, then (b, a) is on f^(-1).

★ MUST MEMORIZE

- **Domain restrictions: denominator != 0; radicand >= 0 (even root); argument of log > 0**
- **(f o g)^(-1) = g^(-1) o f^(-1)**
- **To find inverse: swap x and y, then solve for y**
- **f(f^(-1)(x)) = x and f^(-1)(f(x)) = x (on appropriate domains)**

EXAMPLE — Composition & Inverse

Let f(x) = 2x - 1 and g(x) = x^2 + 3. Find (f o g)(x) and f^(-1)(x).

Solution: (f o g)(x) = f(g(x)) = 2(x^2+3) - 1 = 2x^2 + 5. For inverse: y = 2x-1 → x = (y+1)/2 → f^(-1)(x) = (x+1)/2.

Answer: (f o g)(x) = 2x^2 + 5; f^(-1)(x) = (x+1)/2

UNIT 02

Polynomials & Rational Functions

Zeros, End Behavior, Asymptotes

Core Concepts

Degree n polynomial has at most n real zeros and n-1 turning points.

Factor Theorem: (x - c) is a factor of p(x) iff p(c) = 0.

Rational Root Theorem: possible rational zeros = ±(factors of constant term) / (factors of leading coeff).

Rational function f(x) = p(x)/q(x): vertical asymptote at zeros of q(x) (where p != 0).

Horizontal asymptote: compare degrees. deg p < deg q → y=0; deg p = deg q → y = ratio of leading coeffs; deg p > deg q → oblique (no horizontal).

★ MUST MEMORIZE

- **End behavior: even degree → both ends same direction; odd degree → opposite directions**
- **Multiplicity of zero: odd multiplicity → crosses x-axis; even multiplicity → touches (bounces)**
- **Remainder Theorem: p(c) equals the remainder when p(x) is divided by (x - c)**
- **Complex zeros come in conjugate pairs for real-coefficient polynomials**

EXAMPLE — Finding Zeros

Find all zeros of p(x) = x^3 - 6x^2 + 11x - 6.

Solution: Test x=1: 1-6+11-6=0 ✓. Factor out (x-1): x^3-6x^2+11x-6 = (x-1)(x^2-5x+6) = (x-1)(x-2)(x-3).

Answer: x = 1, 2, 3

UNIT 03

Exponential & Logarithmic Functions

Properties, Equations, Applications

Core Concepts

Exponential: f(x) = a * b^x, b > 0, b != 1. Growth if b > 1; decay if 0 < b < 1.

Natural exponential: e ≈ 2.71828. Compound interest: A = P(1 + r/n)^(nt). Continuous: A = Pe^(rt).

Logarithm: log_b(x) = y means b^y = x. Common log: log = log_10. Natural log: ln = log_e.

Change of base: log_b(x) = ln(x)/ln(b) = log(x)/log(b).

★ MUST MEMORIZE

- **log(xy) = log(x) + log(y) | log(x/y) = log(x) - log(y) | log(x^r) = r*log(x)**
- **b^(log_b(x)) = x and log_b(b^x) = x**
- **To solve: 2^(x+1) = 8 → 2^(x+1) = 2^3 → x+1 = 3 → x = 2**
- **Exponential growth/decay: N(t) = N_0 * e^(kt)**

EXAMPLE — Logarithmic Equation

Solve: log_2(x+3) + log_2(x-1) = 5.

Solution: Combine: log_2[(x+3)(x-1)] = 5 → (x+3)(x-1) = 32 → x^2+2x-3=32 → x^2+2x-35=0 → (x+7)(x-5)=0. x=-7 rejected (log of negative).

Answer: x = 5

UNIT 04

Trigonometric Functions

Unit Circle, Identities, Equations, Graphs

Core Concepts

Unit circle: sin θ = y, cos θ = x, tan θ = y/x. Radius = 1.

Quadrant signs — Q1: all +; Q2: sin +; Q3: tan +; Q4: cos +. Mnemonic: All Students Take Calculus.

Radian-degree: 180° = π rad. Arc length s = rθ. Area of sector = (1/2)r^2 θ.

Graphs: y = A sin(Bx + C) + D. Amplitude |A|, Period = 2π/|B|, Phase shift = -C/B, Vertical shift D.

Sum/Difference: sin(A±B) = sinA cosB ± cosA sinB; cos(A±B) = cosA cosB ■ sinA sinB.

★ MUST MEMORIZE

- **Pythagorean: sin^2θ + cos^2θ = 1 | 1 + tan^2θ = sec^2θ | 1 + cot^2θ = csc^2θ**
- **Double angle: sin(2θ) = 2sinθcosθ | cos(2θ) = cos^2θ - sin^2θ = 1 - 2sin^2θ = 2cos^2θ - 1**
- **Key values: sin30°=1/2, sin45°=√2/2, sin60°=√3/2 (and cosine in reverse order)**
- **Law of Sines: a/sinA = b/sinB = c/sinC | Law of Cosines: a^2 = b^2 + c^2 - 2bc cosA**

EXAMPLE — Trig Identity Proof

Verify: $(\tan x + \cot x) = 1/(\sin x \cos x)$.

Solution: LHS = $\sin x/\cos x + \cos x/\sin x = (\sin^2 x + \cos^2 x)/(\sin x \cos x) = 1/(\sin x \cos x)$ = RHS. ✓

Answer: Identity verified

UNIT 05

Sequences, Series & Conic Sections

Arithmetic, Geometric, Parabola, Ellipse, Hyperbola

Sequences & Series

Arithmetic sequence: $a_n = a_1 + (n-1)d$. Sum $S_n = n/2 * (a_1 + a_n) = n/2 * (2a_1 + (n-1)d)$.

Geometric sequence: $a_n = a_1 * r^{(n-1)}$. Sum $S_n = a_1(1-r^n)/(1-r)$, $r \neq 1$.

Infinite geometric series ($|r| < 1$): $S = a_1/(1-r)$.

Binomial Theorem: $(a+b)^n = \Sigma C(n,k) a^{(n-k)} b^k$, k=0 to n.

Conic Sections

Parabola (vertical): $(x-h)^2 = 4p(y-k)$. Focus at $(h, k+p)$, directrix $y = k-p$.

Ellipse: $(x-h)^2/a^2 + (y-k)^2/b^2 = 1$ ($a>b$). $c^2 = a^2 - b^2$. Foci at $(h\pm c, k)$.

Hyperbola: $(x-h)^2/a^2 - (y-k)^2/b^2 = 1$. $c^2 = a^2 + b^2$. Asymptotes: $y-k = \pm(b/a)(x-h)$.

Circle: $(x-h)^2 + (y-k)^2 = r^2$. Center (h,k), radius r.

★ MUST MEMORIZE

- **Arithmetic: constant difference d. Geometric: constant ratio r**
- **For infinite geometric: must have $|r| < 1$ for sum to exist**
- **Parabola: $p > 0$ opens up/right; $p < 0$ opens down/left**
- **Ellipse: larger denominator determines major axis direction**

EXAMPLE — Geometric Series

Find the sum of the infinite geometric series 12 + 4 + 4/3 + ...

Solution: $a_1 = 12$, $r = 4/12 = 1/3$. $|r| < 1$, so $S = 12/(1 - 1/3) = 12/(2/3) = 18$.

Answer: S = 18

UNIT 06

Limits, Vectors & Matrices

Intro to Calculus, 2D/3D Vectors, Linear Systems

Limits

$\lim_{x \to a} f(x) = L$ means f(x) can be made arbitrarily close to L as $x \to a$.

One-sided limits: left limit $\lim_{x \to a^-}$ and right limit $\lim_{x \to a^+}$. Limit exists iff both sides are equal.

Standard limits: $\lim_{x \to 0} \sin(x)/x = 1$; $\lim_{x \to \infty} (1 + 1/x)^x = e$.

Indeterminate forms (0/0, ∞/∞): factor, rationalize, or apply L'Hôpital's Rule.

Vectors & Matrices

Vector v = (2D) or (3D). Magnitude $|v| = \sqrt{a^2+b^2}$ or $\sqrt{a^2+b^2+c^2}$.

Dot product: $u \cdot v = u_1v_1 + u_2v_2 + \ldots$ Angle: $\cos \theta = (u \cdot v)/(|u||v|)$. Perpendicular: $u \cdot v = 0$.

Cross product (3D): $u \times v$ gives vector perpendicular to both u and v. $|u \times v| = |u||v|\sin \theta$.

Matrix multiplication: $(AB)_{ij} = \Sigma A_{ik}B_{kj}$. Inverse: $AA^{-1} = I$.

★ MUST MEMORIZE

- **Continuity: f is continuous at a if $\lim_{x \to a} f(x) = f(a)$**
- **Unit vector: u-hat = v / |v|**
- **Determinant 2x2: det[[a,b],[c,d]] = ad - bc**
- **System Ax = b: unique solution iff det(A) != 0**

EXAMPLE — Evaluating a Limit

Evaluate $\lim_{x \to 3} (x^2 - 9)/(x - 3)$.

Solution: Factor numerator: (x-3)(x+3)/(x-3) = x+3 (x != 3). As $x \to 3$: 3+3 = 6.

Answer: Limit = 6

PRACTICE EXAM · 20 PROBLEMS

Subjective Format · Show All Work · Exam Style: AP / SAT / IB

Instructions: Solve each problem, showing all steps. Clearly box or underline your final answer. Partial credit may be awarded for correct reasoning. Answers and full solutions begin on the last page.

Q1.

Let $f(x) = 3x + 2$ and $g(x) = x^2 - 1$. Find $(g \circ f)(x)$ and simplify completely. Then determine the domain of $(g \circ f)(x)$.

Q2.

Find the inverse function of $f(x) = (2x - 5)/(x + 1)$. State the domain of $f^{-1}(x)$ and verify by computing $f(f^{-1}(x))$.

Q3.

A polynomial $p(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$ has a known zero at $x = 1$. Use the Factor Theorem and polynomial division to find all remaining zeros, including any complex zeros.

Q4.

Determine all vertical and horizontal asymptotes of the rational function $f(x) = (3x^2 - 12) / (x^2 - x - 6)$. Then sketch a rough description of the graph's behavior near each asymptote.

Q5.

Solve the equation: $2^{3x-1} = 5^{x+2}$. Express your answer in exact form using logarithms, then give a decimal approximation rounded to four decimal places.

Q6.

Solve the system of logarithmic equations: $\log(x) + \log(y) = 3$ $\log(x) - \log(y) = 1$ Find the exact values of x and y.

Q7.

Without a calculator, evaluate: $\sin(195°)$. Show your work using a sum or difference identity. Express your answer in simplified radical form.

Q8.

Prove the trigonometric identity: $\cos^4(x) - \sin^4(x) = \cos(2x)$. Show each algebraic step clearly.

Q9.

Solve for all solutions in $[0, 2\pi)$: $2\cos^2(x) - \cos(x) - 1 = 0$. Give answers in exact radian form.

Q10.

A 40-foot ladder leans against a wall. The foot of the ladder is 10 feet from the base of the wall. Find (a) the angle the ladder makes with the ground (to the nearest tenth of a degree), and (b) how high up the wall the ladder reaches.

Q11.

Find the 50th term and the sum of the first 50 terms of the arithmetic sequence: 7, 13, 19, 25, ...

Q12.

A geometric sequence has its 3rd term equal to 20 and its 6th term equal to 2500/27 ÷ ... Wait — the 3rd term is 20 and the common ratio $r = 5/2$. Find the first term a_1, write the general term a_n, and determine the sum of the first 6 terms.

Q13.

Expand $(2x - 3)^5$ completely using the Binomial Theorem. Identify the coefficient of the x^3 term.

Q14.

Write the equation of the parabola with vertex at (2, -1) that passes through the point (4, 7). State whether it opens up or down, and find the focus and directrix.

Q15.

Find the equation of the ellipse with center at the origin, major axis along the x-axis, a vertex at (5, 0), and a focus at (3, 0). Then find the eccentricity and the equations of the directrices.

Q16.

Evaluate the limit (without L'Hôpital's Rule): $\lim_{x\to4} (x^2 - 16) / (\sqrt{x} - 2)$. Show the algebraic manipulation required.

Q17.

Determine whether the function $f(x) = \{ (x^2 - 4)/(x - 2) \text{ if } x \neq 2, 5 \text{ if } x = 2 \}$ is continuous at $x = 2$. Justify your answer using the definition of continuity.

Q18.

Given vectors $u = \langle 3, -1, 2 \rangle$ and $v = \langle 1, 4, -2 \rangle$, compute: (a) $u \cdot v$, (b) $|u|$ and $|v|$, (c) the angle between u and v (in degrees, rounded to one decimal), and (d) a unit vector in the direction of u.

Q19.

Solve the matrix equation $AX = B$ where $A = [[2, 1],[5, 3]]$ and $B = [[4],[7]]$. Find A^{-1} first, then compute $X = A^{-1}B$.

Q20.

A radioactive substance decays according to $N(t) = N_0 e^{-kt}$. If the initial amount is 500 grams and after 10 years only 350 grams remain, (a) find the decay constant k (exact and approximate), (b) find the half-life $T_{1/2}$, and (c) determine how many grams remain after 30 years.

ANSWER KEY & FULL SOLUTIONS

Detailed step-by-step solutions for all 20 problems

Q1 Answer: (g o f)(x) = 9x^2 + 12x + 3; Domain: all real numbers

→ (g o f)(x) = g(f(x)) = g(3x+2) = (3x+2)^2 - 1

→ = 9x^2 + 12x + 4 - 1 = 9x^2 + 12x + 3

→ Since it is a polynomial, domain = (-inf, +inf)

Q2 Answer: f^(-1)(x) = (-x - 5)/(x - 2); Domain: x != 2

→ Let y = (2x-5)/(x+1). Swap x and y: x = (2y-5)/(y+1)

→ x(y+1) = 2y - 5 → xy + x = 2y - 5 → xy - 2y = -x - 5

→ y(x-2) = -x - 5 → y = (-x-5)/(x-2)

→ Verify: f(f^(-1)(x)) = f((-x-5)/(x-2)) simplifies to x ✓ (Domain: x != 2 since f has range all reals except 2)

Q3 Answer: Zeros: x = 1, 2, 3, -1

→ p(1) = 1 - 5 + 5 + 5 - 6 = 0, so (x-1) is a factor

→ Divide: x^4-5x^3+5x^2+5x-6 ÷ (x-1) = x^3 - 4x^2 + x + 6

→ Test x=2: 8-16+2+6=0 ✓. Divide by (x-2): x^2 - 2x - 3 = (x-3)(x+1)

→ All zeros: x = 1, 2, 3, -1 (all real, no complex zeros)

Q4 Answer: VA: x = 3, x = -2; HA: y = 3

→ Factor: 3x^2-12 = 3(x^2-4) = 3(x-2)(x+2); x^2-x-6 = (x-3)(x+2)

→ Cancel common (x+2): f(x) = 3(x-2)/(x-3), hole at x = -2

→ VA at x = 3 (denominator zero, numerator non-zero)

→ Degrees equal: HA = ratio of leading coefficients = 3/1 = 3, so y = 3

Q5 Answer: x = (2ln5 + ln2)/(3ln2 - ln5) ≈ 5.9469

→ Take ln of both sides: (3x-1)ln2 = (x+2)ln5

→ 3x*ln2 - ln2 = x*ln5 + 2*ln5

→ x(3ln2 - ln5) = 2ln5 + ln2

→ x = (2ln5 + ln2)/(3ln2 - ln5) ≈ (3.2189 + 0.6931)/(2.0794 - 1.6094) ≈ 3.9120/0.4700 ≈ 5.9469 ... (more precisely ≈ 8.3235; recheck: 3ln2≈2.079, ln5≈1.609, 2ln5≈3.219, ln2≈0.693; x = (3.219+0.693)/(2.079-1.609) = 3.912/0.470 ≈ 8.3234)

→ x ≈ 8.3234

Q6 Answer: x = 100, y = 10

→ Add equations: 2log(x) = 4 → log(x) = 2 → x = 100

→ Subtract: 2log(y) = 2 → log(y) = 1 → y = 10

→ Verify: log(100)+log(10)=2+1=3 ✓; log(100)-log(10)=2-1=1 ✓

Q7 Answer: sin(195°) = -(√6 + √2)/4

→ 195° = 150° + 45°

→ sin(150°+45°) = sin150°cos45° + cos150°sin45°

→ = (1/2)(√2/2) + (-√3/2)(√2/2)

→ = √2/4 - √6/4 = (√2 - √6)/4 = -(√6 - √2)/4

Q8 Answer: cos^4x - sin^4x = cos(2x) [Identity Proof]

→ LHS = (cos^2x + sin^2x)(cos^2x - sin^2x) [difference of squares]

→ = (1)(cos^2x - sin^2x) [Pythagorean identity]

→ = cos(2x) [double angle identity] = RHS ✓

Q9 Answer: x = 0, 2pi/3, 4pi/3, pi/3... Wait — let's redo:

→ 2cos^2x - cosx - 1 = 0. Let u = cosx: 2u^2 - u - 1 = (2u+1)(u-1) = 0

→ cosx = -1/2 → x = 2pi/3, 4pi/3

→ cosx = 1 → x = 0

→ Solutions in [0, 2pi): x = 0, 2pi/3, 4pi/3

Q10 Answer: (a) theta ≈ 75.5°; (b) height ≈ 38.7 ft

→ (a) cos(theta) = adj/hyp = 10/40 = 1/4 → theta = arccos(1/4) ≈ 75.5°

→ (b) sin(theta) = h/40 → h = 40 sin(75.5°) ≈ 40(0.9682) ≈ 38.7 ft

→ Alternatively: h = sqrt(40^2 - 10^2) = sqrt(1500) = 10sqrt(15) ≈ 38.73 ft

Q11 Answer: a_50 = 301; S_50 = 7,700

→ d = 13 - 7 = 6; a_n = 7 + (n-1)(6) = 6n + 1

→ a_50 = 6(50) + 1 = 301

→ S_50 = 50/2 * (7 + 301) = 25 * 308 = 7,700

Q12 Answer: a_1 = 32/5; a_n = (32/5)(5/2)^(n-1); S_6 = 3906/5 = 781.2

→ a_3 = a_1 * r^2 = a_1 * (5/2)^2 = (25/4)a_1 = 20 → a_1 = 80/25 = 16/5

→ Wait: let's recompute — a_3 = 20, r = 5/2: a_1 = 20/(5/2)^2 = 20/(25/4) = 80/25 = 16/5

→ a_n = (16/5)(5/2)^(n-1)

→ S_6 = (16/5)(1-(5/2)^6)/(1-5/2) = (16/5)(1 - 15625/64)/(-3/2)

→ = (16/5)(-15561/64)/(-3/2) = (16/5)(15561/64)(2/3) = (16*2*15561)/(5*64*3) = 497952/960 = 518.7

→ More precisely: S_6 = a_1(r^6-1)/(r-1) = (16/5)((5/2)^6-1)/(3/2) = (16/5)(15561/64)(2/3) ≈ 518.7

Q13 Answer: Coefficient of x^3 is -720

→ (2x-3)^5 = Σ C(5,k)(2x)^(5-k)(-3)^k for k=0 to 5

→ For x^3 term: 5-k=3 → k=2

→ C(5,2)(2x)^3(-3)^2 = 10 * 8x^3 * 9 = 720x^3

→ Full expansion: 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243

→ Coefficient of x^3 = 720

Q14 Answer: y = 2(x-2)^2 - 1; Opens up; Focus: (2, -3/8+...)

→ Vertex (2,-1), passes (4,7): 7 = a(4-2)^2 - 1 → 8 = 4a → a = 2

→ Equation: y = 2(x-2)^2 - 1. Opens UPWARD (a > 0).

→ Rewrite: (x-2)^2 = (1/2)(y+1), so 4p = 1/2 → p = 1/8

→ Focus: (2, -1 + 1/8) = (2, -7/8). Directrix: y = -1 - 1/8 = -9/8

Q15 Answer: x^2/25 + y^2/16 = 1; e = 3/5; directrices x = ±25/3

→ a = 5, c = 3 → b^2 = a^2 - c^2 = 25 - 9 = 16

→ Equation: x^2/25 + y^2/16 = 1

→ Eccentricity: e = c/a = 3/5

→ Directrices: x = ±a/e = ±5/(3/5) = ±25/3

Q16 Answer: Limit = 16

→ Rationalize: multiply numerator and denominator by (√x + 2)

→ [(x^2-16)(√x+2)] / [(√x-2)(√x+2)] = (x^2-16)(√x+2)/(x-4)

→ = (x-4)(x+4)(√x+2)/(x-4) = (x+4)(√x+2) for x != 4

→ As x→4: (4+4)(√4+2) = 8*4 = 32 ... wait: (x^2-16)/(√x-2)

→ Substitute x=4: (16-16)/(2-2)=0/0. Rationalize numerator: (x^2-16)=(x-4)(x+4)

→ Multiply by (√x+2)/(√x+2): (x-4)(x+4)(√x+2)/(x-4) = (x+4)(√x+2) → (8)(4) = 32

→ Limit = 32

Q17 Answer: f is NOT continuous at x = 2 (limit = 4, but f(2) = 5)

→ Step 1: f(2) = 5 (given)

→ Step 2: lim_{x→2} (x^2-4)/(x-2) = lim_{x→2} (x-2)(x+2)/(x-2) = lim_{x→2}(x+2) = 4

→ Step 3: lim_{x→2} f(x) = 4 ≠ 5 = f(2)

→ Since the limit (4) ≠ f(2) (5), f is DISCONTINUOUS at x = 2 (removable discontinuity)

Q18 Answer: (a) u·v = -5; (b) |u|=√14, |v|=√21; (c) ≈ 106.8°; (d) <3/√14, -1/√14, 2/√14>

→ (a) u·v = 3(1) + (-1)(4) + 2(-2) = 3 - 4 - 4 = -5

→ (b) |u| = √(9+1+4) = √14 ≈ 3.742; |v| = √(1+16+4) = √21 ≈ 4.583

→ (c) cos θ = -5/(√14·√21) = -5/√294 ≈ -5/17.146 ≈ -0.2916 → θ ≈ 106.9°

→ (d) u-hat = (1/√14)<3,-1,2> = <3√14/14, -√14/14, 2√14/14>

Q19 Answer: X = [[5], [-6]]

→ A^(-1) = (1/det(A)) * [[3,-1],[-5,2]]. det(A) = 2*3 - 1*5 = 6-5 = 1

→ So A^(-1) = [[3,-1],[-5,2]]

→ X = A^(-1)B = [[3,-1],[-5,2]] * [[4],[7]] = [[3*4+(-1)*7],[(-5)*4+2*7]] = [[12-7],[-20+14]] = [[5],[-6]]

→ Solution: x_1 = 5, x_2 = -6

Q20 Answer: (a) $k = -\ln(0.7)/10 \approx 0.03567$; (b) $T_{1/2} = \ln 2/k \approx 19.43$ yrs; (c) ≈ 171.5 g

→ (a) $350 = 500 e^{-10k}$ → $e^{-10k} = 0.7$ → $-10k = \ln(0.7)$ → $k = -\ln(0.7)/10 \approx 0.03567$

→ (b) Half-life: $250 = 500 e^{-kt}$ → $-kt = \ln(0.5)$ → $t = \ln 2/k = \ln 2/0.03567 \approx 19.43$ years

→ (c) $N(30) = 500 e^{-0.03567 \cdot 30} = 500 e^{-1.0701} \approx 500(0.3430) \approx 171.5$ grams

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— End of Pre-Calculus Master Guide —